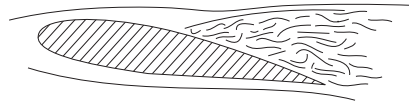# 8

Viscous Incompressible Flows

8.1 INTRODUCTION

In the analysis of motion of a real fluid, the effect of viscosity should be given consideration. Influence of viscosity is more pronounced near the boundary of a solid body immersed in a fluid in motion. The relationship between stress and rate of strain for the motion of real fluid flow was first put forward by Sir Isaac Newton and for this reason the viscosity law bears his name. Later on, G.G. Stokes, an English mathematician and C.L.M.H. Navier, a French engineer, derived the exact equations that govern the motion of real fluids. These equations are in general valid for compressible or incompressible laminar flows and known as Navier-Stokes equations. When a motion becomes turbulent, these equations are generally not able to provide with a complete solution. Usually, in order to obtain accurate results for such situations, the Navier-Stokes equations are modified and solved based on several semi-empirical theories. In the recent past, some researchers have proposed that, on a fine enough scale, all turbulent flows obey the Navier-Stokes equation and computationally, if a fine enough grid is used with appropriate discretization methods, may be both the fine scale and large scale aspects of turbulence can be captured. However, in this chapter we shall discuss the equation of motion for laminar flows and various other aspects of laminar incompressible flows.

8.2 GENERAL VISCOSITY LAW

The well-known Newton's viscosity law is

$$\tau = \mu \frac{\partial V}{\partial n} \tag{8.1}$$

where n is the coordinate direction normal to the solid-fluid interface, μ is the coefficient of viscosity and V is velocity. This law is valid for parallel flows. There are more generalized relations which can relate stress field and velocity field for any kind of flow. Such relations are called *constitutive equations*. We shall consider here the Stokes' viscosity law.

According to Stokes' law of viscosity, shear stress is proportional to rate of shear strain so that

$$\tau_{xy} = \tau_{yx} = \mu \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right] \tag{8.2a}$$

$$\tau_{yz} = \tau_{zy} = \mu \left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right] \tag{8.2b}$$

$$\tau_{zx} = \tau_{xz} = \mu \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right] \tag{8.2c}$$

The first subscript of τ denotes the direction of the normal to the plane on which the stress acts, while the second subscript denotes direction of the force which causes the stress.

The expressions of Stokes' law of viscosity for normal stresses are

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} + \mu' \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] \tag{8.3a}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y} + \mu' \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] \tag{8.3b}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z} + \mu' \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] \tag{8.3c}$$

where μ' is related to the second coefficient of viscosity μ_1 by the relationship $\mu' = -\frac{2}{3} (\mu - \mu_1)$. We have already seen that thermodynamic pressure $p = -\frac{(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})}{3}$. Now, if we add the three Eqs (8.3a), (8.3b) and (8.3c), we obtain

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3p + 2\mu \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] + 3\mu' \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right]$$

or $\quad \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3p + (2\mu + 3\mu') \left[\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} \right] \tag{8.4}$

For incompressible fluid, $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = \nabla \cdot \vec{V} = 0$

So, $p = -\dfrac{(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})}{3}$ is satisfied in the same manner. For compressible

fluids, Stokes' hypothesis is $\mu' = -\dfrac{2}{3}\mu$. Invoking this to Eq. (8.4), will finally

conclude that $p = -\dfrac{(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})}{3}$. Generally, fluids obeying the ideal gas

equation follow this hypothesis and they are called Stokesian fluids. It may also be mentioned that the second coefficient of viscosity, μ_1, has been verified to be negligibly small.

Now, we can write

$$\sigma_{xx} = -p + 2\mu\,\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] \qquad (8.5a)$$

$$\sigma_{yy} = -p + 2\mu\,\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] \qquad (8.5b)$$

$$\sigma_{zz} = -p + 2\mu\,\frac{\partial w}{\partial z} - \frac{2}{3}\mu\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] \qquad (8.5c)$$

In deriving the above stress-strain rate relationship, it was assumed that a fluid has the following properties

1. Fluid is homogeneous and isotropic, i.e. the relation between components of stress and those of rate of strain is the same in all directions.
2. Stress is a linear function of strain rate.
3. The stress-strain relationship will hold good irrespective of the orientation of the reference coordinate system.
4. The stress components must reduce to the hydrostatic pressure (typically thermodynamic pressure = hydrostatic pressure) p when all the gradients of velocities are zero.

8.3   NAVIER–STOKES EQUATIONS

Generalized equations of motion of a real flow are named after the inventors of them and they are known as Navier–Stokes equations. However, they are derived from the Newton's second law which states that the product of mass and acceleration is equal to sum of the external forces acting on a body. External forces are of two kinds—one acts throughout the mass of the body and another acts on the boundary.

The first one is known as body force (gravitational force, electromagnetic force) and the second one is surface force (pressure and frictional force).

Let the body force per unit mass be

$$\vec{f}_b = \hat{i}\, f_x + \hat{j}\, f_y + \hat{k}\, f_z \tag{8.6}$$

and surface force per unit volume be

$$\vec{F} = \hat{i}\, F_x + \hat{j}\, F_y + \hat{k}\, F_z \tag{8.7}$$

Consider a differential fluid element in the flow field (Fig. 8.1). We wish to evaluate the surface forces acting on the boundary of this rectangular parallel-epiped.
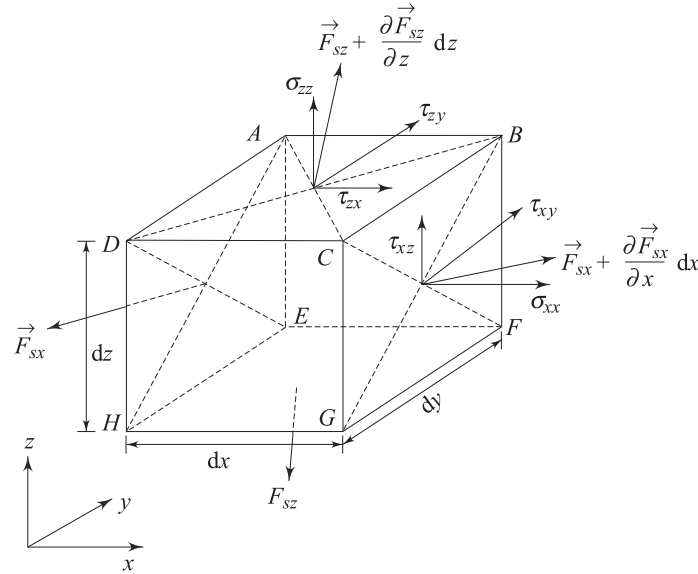

Fig. 8.1 Definition of the components of stress and their locations in a differential fluid element

To accomplish this, we shall consider surface force on the surface *AEHD*, per unit area,

$$\hat{i}\, \sigma_{xx} + \hat{j}\, \tau_{xy} + \hat{k}\, \tau_{xz} = \vec{F}_{sx}$$

Surface force on the surface *BFGC* per unit area is

$$\vec{F}_{sx} + \frac{\partial \vec{F}_{sx}}{\partial x}\, dx$$

Net force on the body due to imbalance of surface forces on the above two surfaces is

$$\frac{\partial \vec{F}_{sx}}{\partial x}\, dx\, dy\, dz \tag{8.8}$$

Total force on the body due to net surface forces on all six surfaces is

$$\left(\frac{\partial \vec{F}_{sx}}{\partial x} + \frac{\partial \vec{F}_{sy}}{\partial y} + \frac{\partial \vec{F}_{sz}}{\partial z} \right) dx\, dy\, dz \tag{8.9}$$

and the resultant surface force dF per unit volume, is

$$d\vec{F} = \frac{\partial \vec{F}_{sx}}{\partial x} + \frac{\partial \vec{F}_{sy}}{\partial y} + \frac{\partial \vec{F}_{sz}}{\partial z} \tag{8.10}$$

The quantities $\vec{F}_{sx}$, $\vec{F}_{sy}$ and $\vec{F}_{sz}$ are vectors which can be resolved into normal stresses denoted by σ and shearing stresses denoted by τ as

$$\left. \begin{aligned}
\vec{F}_{sx} &= \hat{i}\,\sigma_{xx} + \hat{j}\,\tau_{xy} + \hat{k}\,\tau_{xz} \\
\vec{F}_{sy} &= \hat{i}\,\tau_{yx} + \hat{j}\,\sigma_{yy} + \hat{k}\,\tau_{yz} \\
\vec{F}_{sz} &= \hat{i}\,\tau_{zx} + \hat{j}\,\tau_{zy} + \hat{k}\,\sigma_{zz}
\end{aligned} \right\} \tag{8.11}$$

The stress system is having nine scalar quantities. These nine quantities form a stress tensor. The set of nine components of stress tensor can be described as

$$\pi = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{8.12}$$

The above stress tensor is symmetric, which means that two shearing stresses with subscripts which differ only in their sequence are equal. Considering the equation of motion for instantaneous rotation of the fluid element (Fig. 8.1) about y axis, we can write

$$\dot{\omega}_y\, dI_y = (\tau_{xz}\, dy\, dz)dx - (\tau_{zx}\, dx\, dy)dz$$
$$= (\tau_{xz} - \tau_{zx})\, d\forall$$

where d$\forall$ is the volume of the element, and $\dot{\omega}_y$ and dI_y are the angular acceleration and moment of inertia of the element about y-axis respectively. Since dI_y is proportional to fifth power of the linear dimensions and d$\forall$ is proportional to the third power of the linear dimensions, the left hand side of the above equation vanishes faster than the right hand side on contracting the element to a point. Hence, the result is

$$\tau_{xz} = \tau_{zx}$$

From the similar considerations about other two remaining axes, we can write

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{yz} = \tau_{zy}$$

which has already been observed in Eqs (8.2a), (8.2b) and (8.2c) earlier.

Invoking these conditions into Eq. (8.12), the stress tensor becomes

$$\pi = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \tag{8.13}$$

Combining Eqs (8.10), (8.11) and (8.13), the resultant surface force per unit volume becomes

$$d\vec{F} = \hat{i}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right)$$

$$+ \hat{j}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right)$$

$$+ \hat{k}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \tag{8.14}$$

As per the velocity field,

$$\frac{D\vec{V}}{Dt} = \hat{i}\frac{Du}{Dt} + \hat{j}\frac{Dv}{Dt} + \hat{k}\frac{Dw}{Dt} \tag{8.15}$$

By Newton's law of motion applied to the differential element, we can write

$$\rho(dx\,dy\,dz)\frac{D\vec{V}}{Dt} = (d\vec{F})\,(dx\,dy\,dz) + \rho\vec{f}_b\,(dx\,dy\,dz)$$

or

$$\rho\frac{D\vec{V}}{Dt} = d\vec{F} + \rho\vec{f}_b$$

Substituting Eqs (8.15), (8.14) and (8.6) into the above expression, we obtain

$$\rho\frac{Du}{Dt} = \rho f_x + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) \tag{8.16a}$$

$$\rho\frac{Dv}{Dt} = \rho f_y + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) \tag{8.16b}$$

$$\rho\frac{Dw}{Dt} = \rho f_z + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \tag{8.16c}$$

In order to express $\frac{Du}{Dt}$, $\frac{Dv}{Dt}$ and $\frac{Dw}{Dt}$ in terms of field derivatives, Eqs (8.2) and (8.5) are introduced into Eq. (8.16) and we obtain

$$\rho\frac{Du}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V} \right) \right]$$

$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] \tag{8.17a}$$

$$\rho \, \frac{Dv}{Dt} \;=\; \rho f_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla \cdot \vec{V} \right)\right]$$

$$+ \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)\right] \qquad (8.17b)$$

and $\quad \rho \, \dfrac{Dw}{Dt} \;=\; \rho f_z - \dfrac{\partial p}{\partial z} + \dfrac{\partial}{\partial z}\left[\mu\left(2\dfrac{\partial w}{\partial z} - \dfrac{2}{3}\nabla \cdot \vec{V} \right)\right]$

$$+ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)\right] \qquad (8.17c)$$

These differential equations are known as Navier–Stokes equations. At this juncture, it is necessary to discuss the equation of continuity as well, which is having a general form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad (8.18)$$

The general form of continuity equation is simplified in case of incompressible flow where ρ = constant. Equation of continuity for incompressible flow becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (8.19)$$

Invoking Eq. (8.19) into Eqs (8.17a), (8.17b) and (8.17c), we get

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right)$$

$$= \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \qquad (8.20a)$$

$$\rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right)$$

$$= \rho f_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \qquad (8.20b)$$

$$\rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right)$$

$$= \rho f_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \qquad (8.20c)$$

In short, vector notation may be used to write Navier-Stokes and continuity equations for incompressible flow as

$$\rho \, \frac{D\vec{V}}{Dt} = \rho \vec{f}_b - \nabla p + \mu \nabla^2 \vec{V} \qquad (8.21)$$

and $\qquad\qquad \nabla \cdot \vec{V} = 0 \qquad (8.22)$

We observe that we have four unknown quantities, u, v, w and p, and four equations,—equations of motion in three directions and the continuity equation. In principle, these equations are solvable but to date generalized solution is not available due to the complex nature of the set of these equations. The highest order terms, which come from the viscous forces, are linear and of second order. The first order convective terms are non-linear and hence, the set is termed as quasi-linear.

Navier–Stokes equations in cylindrical coordinate (Fig. 8.2) are useful in solving many problems. If v_r, v_θ and v_z denote the velocity components along the radial, cross-radial and axial directions respectively, then for the case of incompressible flow, Eqs (8.21) and (8.22) lead to the following system of equations:
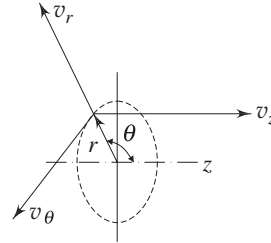

Fig. 8.2   Cyl+indrical polar coordinate and the velocity components

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\cdot\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = \rho f_r - \frac{\partial p}{\partial r}$$

$$+ \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r^2}\cdot\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\cdot\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right) \quad (8.23a)$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cdot\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r\, v_\theta}{r}\right) = \rho f_\theta - \frac{1}{r}\cdot\frac{\partial p}{\partial \theta}$$

$$+ \mu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} + \frac{1}{r^2}\cdot\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\cdot\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right) \quad (8.23b)$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\cdot\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$

$$= \rho f_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) \quad (8.23c)$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\cdot\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \quad (8.24)$$

Let us quickly look at a little more general way of deriving the Navier-Stokes equations from the basic laws of physics. Consider a general flow field as represented in Fig. 8.3. Let us imagine a closed control volume $¥_0$ within the

flow field. A control surface, A_0, is defined as the surface which bounds the volume V_0. The control volume is fixed in space and the fluid is moving through it. The control volume occupies reasonably large finite region of the flow field.

According to Reynolds transport theorem, we know that the laws of physics which are basically stated for a system, can be re-stated for a control volume through some integral relationship. However, for momentum conservation, the Reynolds transport theorem states, "The rate of change of momentum for a system equals the sum of the rate of change of momentum inside the control volume and the rate of efflux of momentum across the control surface."

Again, the rate of change of momentum for a system (in our case the control volume is the system) is equal to the net external force acting on it. Now, we shall transform these statements into equation by accounting for each term.
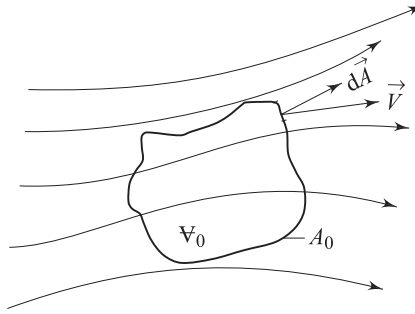

Fig. 8.3 Finite control volume fixed in space with the fluid moving through it

Rate of change of momentum inside the control volume

$$= \frac{\partial}{\partial t} \iiint_{V_0} \rho \vec{V} \, dV$$

$$= \iiint_{V_0} \frac{\partial}{\partial t} (\rho \vec{V}) \, dV \quad \text{(since } t \text{ is independent of space variable)} \qquad (8.25)$$

Rate of efflux of momentum through control surface

$$= \iint_{A_0} \rho \vec{V} (\vec{V} \cdot d\vec{A}) = \iint_{A_0} \rho \vec{V} \vec{V} \cdot \vec{n} \, dA$$

$$= \iiint_{V_0} \nabla \cdot (\rho \vec{V} \vec{V}) \, dV$$

$$= \iiint_{V_0} \left(\vec{V} (\nabla \cdot \rho \vec{V}) + \rho \vec{V} \cdot \nabla \vec{V} \right) dV \qquad (8.26)$$

Surface force acting on the control volume

$$= \iint_{A_0} d\vec{A} \cdot \sigma$$

$$= \iint\limits_{A_0} \sigma \cdot \mathrm{d}\vec{A} \quad [\sigma \text{ is symmetric stress tensor}]$$

$$= \iiint\limits_{\Psi_0} (\nabla \cdot \sigma) \mathrm{d}\Psi \tag{8.27}$$

and, the body force acting on the control volume

$$= \iiint\limits_{\Psi_0} \rho \vec{f} \, \mathrm{d}\Psi \tag{8.28}$$

$\vec{f}$ in Eq. (8.28) is the body force per unit mass.

Finally, we get,

$$\iiint\limits_{\Psi_0} \left(\frac{\partial}{\partial t}(\rho \vec{V}) + \{\vec{V}(\nabla \cdot \rho \vec{V}) + \rho \vec{V} \cdot \nabla \vec{V}\} \right) \mathrm{d}\Psi$$

$$= \iiint\limits_{\Psi_0} (\nabla \cdot \sigma + \rho \vec{f}) \, \mathrm{d}\Psi$$

or

$$\rho \frac{\partial \vec{V}}{\partial t} + \vec{V} \frac{\partial \rho}{\partial t} + \rho \vec{V} \cdot \nabla \vec{V} + \vec{V}(\nabla \cdot \rho \vec{V}) = \nabla \cdot \sigma + \rho \vec{f}$$

or

$$\rho \left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right) + \vec{V} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} \right) = \nabla \cdot \sigma + \rho \vec{f} \tag{8.29}$$

We know that $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$ is the general form of mass conservation equation, valid for both compressible and incompressible flows. Invoking this relationship in Eq. (8.29), we obtain

$$\rho \left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right) = \nabla \cdot \sigma + \rho \vec{f}$$

or

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \nabla \cdot \sigma + \rho \vec{f} \tag{8.30}$$

Equation (8.30) is often referred to as Cauchy's equation of motion. In this equation, σ, the stress tensor, is given by

$$\sigma = -p\vec{I}^* + \mu' \left(\nabla \cdot \vec{V} \right) + 2\mu \, (\text{Def } \vec{V})^{**} \tag{8.31}$$

$$*I \qquad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$** (\text{Def } \vec{V}) = \frac{1}{2} \left(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j} \right)$$

From Eq. (8.31), we get

$$\nabla \cdot \sigma = -\nabla p + (\mu' + \mu)\ \nabla\left(\nabla \cdot \vec{V}\right) + \mu\, \nabla^2 \vec{V} \tag{8.32}$$

and also from *Stokes's* hypothesis, $\mu' + \dfrac{2}{3}\mu = 0$ (8.33)

Invoking Eq. (8.32) in Eq. (8.30) and introducing Eq. (8.33) will yield

$$\rho\frac{D\vec{V}}{Dt} = -\nabla p + \mu\nabla^2\vec{V} + \frac{1}{3}\mu\nabla(\nabla \cdot \vec{V}) + \rho\vec{f} \tag{8.34}$$

This is the most general form of Navier-Stokes equation. The specific forms for different coordinate systems can easily be obtained from Eq. (8.34).

In a cartesian coordinate system,

$$(\text{Def }\vec{V}) = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right) \\[3mm] \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right) \\[3mm] \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right) & \dfrac{\partial w}{\partial z} \end{bmatrix}$$

This is known as deformation tensor. It can be readily seen that (Def $\vec{V}$) is a symmetric tensor.

8.4 EXACT SOLUTIONS OF NAVIER–STOKES EQUATIONS

The basic difficulty in solving Navier-Stokes equations arises due to the presence of nonlinear (quadratic) inertia terms on the left hand side. However, there are some nontrivial solutions of the Navier-Stokes equations in which the nonlinear inertia terms are identically zero. One such class of flows is termed as parallel flows in which only one velocity term is nontrivial and all the fluid particles move in one direction only.

Let us choose x to be the direction along which all fluid particles travel, i.e. $u \neq 0$, $v = w = 0$. Invoking this in continuity equation, we get

$$\frac{\partial u}{\partial x} + \frac{\partial v^{0}}{\partial y} + \frac{\partial w^{0}}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = 0 \qquad \text{which means } u = u\,(y, z, t)$$

Now, Navier-Stokes equations for incompressible flow become

$$\frac{\partial u}{\partial t} + u\frac{\partial u^{0}}{\partial x} + v\frac{\partial u^{0}}{\partial y} + w\frac{\partial u^{0}}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\left[\frac{\partial^2 u^{0}}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\frac{\partial \cancel{v}^0}{\partial \cancel{t}} + u \frac{\partial \cancel{v}^0}{\partial \cancel{x}} + v \frac{\partial \cancel{v}^0}{\partial \cancel{y}} + w \frac{\partial \cancel{v}^0}{\partial \cancel{z}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + v\left[\frac{\partial^2 \cancel{v}^0}{\partial \cancel{x}^2} + \frac{\partial^2 \cancel{v}^0}{\partial \cancel{y}^2} + \frac{\partial^2 \cancel{v}^0}{\partial \cancel{z}^2}\right]$$

$$\frac{\partial w_{\cancel{t}}^0}{\partial \cancel{t}} + u \frac{\partial w_{\cancel{x}}^0}{\partial \cancel{x}} + v \frac{\partial w_{\cancel{y}}^0}{\partial \cancel{y}} + w \frac{\partial w_{\cancel{z}}^0}{\partial \cancel{z}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + v\left[\frac{\partial^2 w_{\cancel{x}}^0}{\partial \cancel{x}^2} + \frac{\partial^2 w^0}{\partial \cancel{y}^2} + \frac{\partial^2 w_{\cancel{z}}^0}{\partial \cancel{z}^2}\right]$$

So, we obtain

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0, \text{ which means } p = p(x) \text{ alone,}$$

and
$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\cdot\frac{\partial p}{\partial x} + v\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \tag{8.35}$$

For a steady two dimensional flow through parallel plates, Eq. (8.35) is further simplified and analytical solution can be obtained. An insightful description of many such analytical solutions in different geometrical configurations has been well documented in White [1] and Faber [2]. However, we shall discuss some of the important exact solutions in the following sections.

8.4.1 Parallel Flow in a Straight Channel

Consider steady flow between two infinitely broad parallel plates as shown in Fig. 8.4. Flow is independent of any variation in z direction, hence, z dependence is gotten rid of and Eq. (8.35) becomes
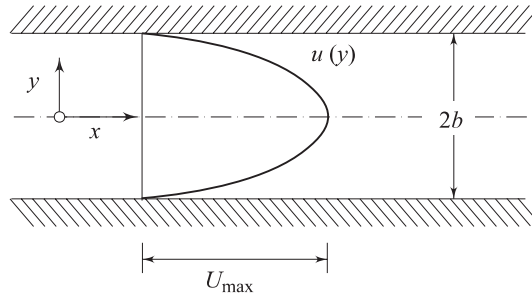

Fig. 8.4 Parallel flow in a straight channel

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \mu\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} \tag{8.36}$$

The boundary conditions are at $y = b$, $u = 0$; and $y = -b$, $u = 0$.
From Eq. (8.36), we can write

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\, y + C_1$$

or
$$u = \frac{1}{2\mu}\cdot\frac{\mathrm{d}p}{\mathrm{d}x}\, y^2 + C_1 y + C_2$$

Applying the boundary conditions, the constants are evaluated as:

$$C_1 = 0 \quad \text{and} \quad C_2 = -\frac{1}{\mu} \cdot \frac{dp}{dx} \cdot \frac{b^2}{2}$$

So, the solution is

$$u = -\frac{1}{2\mu} \cdot \frac{dp}{dx}(b^2 - y^2) \tag{8.37}$$

which implies that the velocity profile is parabolic. We can extend our analysis little further in order to establish the relationship between the maximum velocity and average velocity in the channel.

At $y = 0$, $\quad u = U_{max}$; this yields

$$U_{max} = -\frac{b^2}{2\mu} \cdot \frac{dp}{dx} \tag{8.38a}$$

On the other hand, the average velocity,

$$U_{av} = \frac{Q}{2b} = \frac{\text{flow rate}}{\text{flow area}} = \frac{1}{2b}\int_{-b}^{b} u \, dy$$

or

$$U_{av} = \frac{2}{2b}\int_{0}^{b} u \, dy = \frac{1}{b}\int_{0}^{b} -\frac{1}{2\mu}\cdot\frac{dp}{dx}(b^2 - y^2)\,dy$$

$$= -\frac{1}{2\mu}\frac{dp}{dx}\cdot\frac{1}{b}\left\{\left[b^2 y\right]_0^b - \left(\frac{y^3}{3}\right)_0^b\right\}$$

Finally,

$$U_{av} = -\frac{1}{2\mu}\frac{dp}{dx}\frac{2}{3}b^2 \tag{8.38b}$$

So,

$$\frac{U_{av}}{U_{max}} = \frac{2}{3} \quad \text{or} \quad U_{max} = \frac{3}{2}U_{av} \tag{8.38c}$$

The shearing stress at the wall for the parallel flow in a channel can be determined from the velocity gradient as

$$\tau_{yx}\big|_b = \mu\left(\frac{\partial u}{\partial y}\right)_b = b\frac{dp}{dx} = -2\mu\frac{U_{max}}{b}$$

Since the upper plate is a "minus y surface", a negative stress acts in the positive x direction, i.e. to the right.

The local friction coefficient, C_f, is defined by

$$C_f = \frac{|(\tau_{yx})_b|}{\frac{1}{2}\rho U_{av}^2} = \frac{3\mu\, U_{av}/b}{\frac{1}{2}\rho U_{av}^2}$$

or

$$C_f = \frac{12}{\dfrac{\rho U_{av}(2b)}{\mu}} = \frac{12}{\text{Re}} \tag{8.38d}$$

where Re $= U_{av}(2b)/v$ is the Reynolds number of flow based on average velocity and the channel height ($2b$). Experiments show that Eq. (8.38d) is valid in the laminar regime of the channel flow. The maximum Reynolds number value corresponding to fully developed laminar flow, for which a stable motion will persist, is 2300. In a reasonably careful experiment, laminar flow can be observed up to even Re $= 10,000$. But the value below which the flow will always remain laminar, i.e. the critical value of Re is 2300.

8.4.2 Couette Flow

Another simple solution for Eq. (8.35) is obtained for Couette flow between two parallel plates (Fig. 8.5). Here, one plate is at rest and the other is moving with a velocity U. Let us assume the plates are infinitely large in z direction, so the z dependence is not there and the governing equation is $\dfrac{dp}{dx} = \mu \dfrac{d^2u}{dy^2}$ subjected to the boundary conditions at $y = 0$, $u = 0$ and $y = h$, $u = U$.
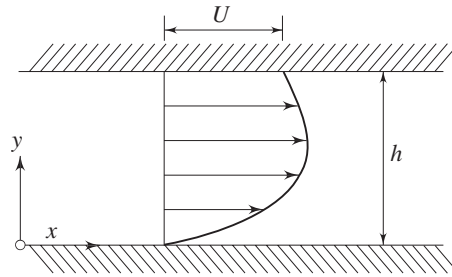

Fig. 8.5 Couette flow between two parallel flat plates

We get,

$$u = \frac{1}{2\mu} \cdot \frac{dp}{dx} y^2 + C_1 y + C_2$$

Invoking the condition (at $y = 0$, $u = 0$), C_2 becomes equal to zero.

$$u = \frac{1}{2\mu} \cdot \frac{dp}{dx} y^2 + C_1 y$$

Invoking the other condition (at $y = h$, $u = U$),

$$C_1 = \frac{U}{h} - \frac{1}{2\mu} \cdot \frac{dp}{dx} h$$

So,
$$u = \frac{y}{h} U - \frac{h^2}{2\mu} \cdot \frac{dp}{dx} \cdot \frac{y}{h} \left(1 - \frac{y}{h}\right) \tag{8.39}$$

Equation (8.39) can also be expressed in the form

$$\frac{u}{U} = \frac{y}{h} - \frac{h^2}{2\mu U} \cdot \frac{dp}{dx} \cdot \frac{y}{h} \left(1 - \frac{y}{h}\right)$$

or
$$\frac{u}{U} = \frac{y}{h} + P\frac{y}{h}\left(1 - \frac{y}{h}\right)$$
(8.40a)

where
$$P = -\frac{h^2}{2\mu U}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)$$

Equation (8.40a) describes the velocity distribution in non-dimensional form across the channel with P as a parameter known as the non-dimensional pressure gradient. When $P = 0$, the velocity distribution across the channel is reduced to

$$\frac{u}{U} = \frac{y}{h}$$

This particular case is known as simple Couette flow. When $P > 0$, i.e. for a *negative* or *favourable* pressure gradient $(-\,\mathrm{d}p/\mathrm{d}x)$ in the direction of motion, the velocity is positive over the whole gap between the channel walls. For negative value of P ($P < 0$), there is a *positive* or *adverse* pressure gradient in the direction of motion and the velocity over a portion of channel width can become negative and back flow may occur near the wall which is at rest. Figure 8.6a shows the effect of dragging action of the upper plate exerted on the fluid particles in the channel for different values of pressure gradient.
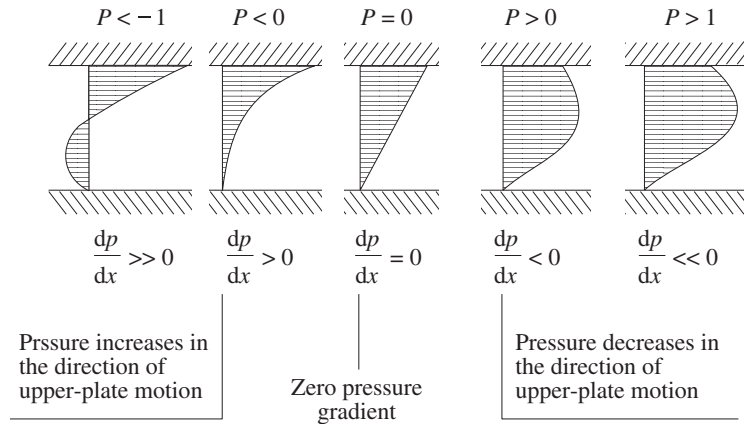

$P < -1$	$P < 0$	$P = 0$	$P > 0$	$P > 1$
$\dfrac{\mathrm{d}p}{\mathrm{d}x} \gg 0$	$\dfrac{\mathrm{d}p}{\mathrm{d}x} > 0$	$\dfrac{\mathrm{d}p}{\mathrm{d}x} = 0$	$\dfrac{\mathrm{d}p}{\mathrm{d}x} < 0$	$\dfrac{\mathrm{d}p}{\mathrm{d}x} \ll 0$

Prssure increases in the direction of upper-plate motion   Zero pressure gradient   Pressure decreases in the direction of upper-plate motion

Fig. 8.6a   Velocity profile for the Couette flow for various values of pressure gradient

Maximum and Minimum Velocities   The quantitative description of non-dimensional velocity distribution across the channel, depicted by Eq. (8.40a), is shown in Fig. 8.6b. The location of maximum or minimum velocity in the channel is found out by setting the derivative $\mathrm{d}u/\mathrm{d}y$ equal to zero. From Eq. (8.40a), we can write

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{U}{h} + \frac{PU}{h}\left(1 - 2\frac{y}{h}\right)$$

For maximum or minimum velocity,

$$\frac{du}{dy} = 0$$

which gives
$$\frac{y}{h} = \frac{1}{2} + \frac{1}{2P} \tag{8.40b}$$

It is interesting to note that maximum velocity for $P = 1$ occurs at $y/h = 1$ and equals to U. For $P > 1$, the maximum velocity occurs at a location $\frac{y}{h} < 1$. This means that with $P > 1$, the fluid particles attain a velocity higher than that of the moving plate at a location somewhere below the moving plate. On the other hand, when $P = -1$, the minimum velocity occurs, according to Eq. (8.40b), at $\frac{y}{h} = 0$. For $P < -1$, the minimum velocity occurs at a location $\frac{y}{h} > 0$. This means that there occurs a back flow near the fixed plate. The values of maximum and minimum velocities can be determined by substituting the value of y from Eq. (8.40b) into Eq. (8.40a) as

$$u_{\max} = \frac{U(1 + P)^2}{4P} \qquad \text{for } P \ge 1$$

$$u_{\min} = \frac{U(1 + P)^2}{4P} \qquad \text{for } P \le 1 \tag{8.40c}$$
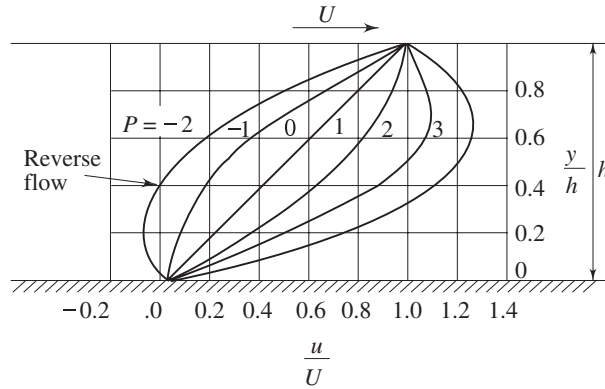

Fig. 8.6b Velocity distribution of the Couette flow

8.4.3 Hagen Poiseuille Flow

Consider *fully developed* laminar flow through a straight tube of circular cross-section as in Fig. 8.7. Rotational symmetry is considered to make the flow two-dimensional axisymmetric. Let us take z-axis as the axis of the tube along which all the fluid particles travel, i.e.
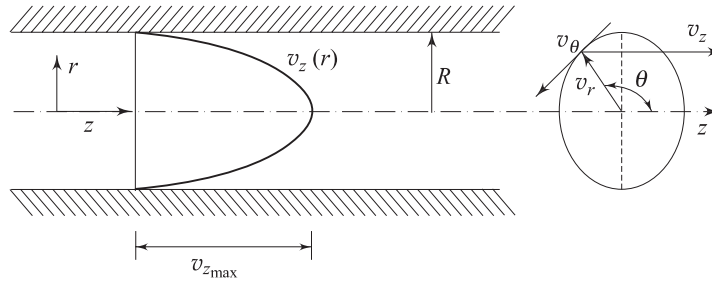
$$v_z \ne 0, \ v_r = 0, \ v_\theta = 0$$

Fig. 8.7 Hagen–Poiseuille flow through a pipe

Now, from continuity equation, we obtain

$$\frac{\partial v_r^0}{\partial r} + \frac{v_r^0}{r} + \frac{\partial v_z}{\partial z} = 0 \quad \left[\text{For rotational symmetry, } \frac{1}{r} \cdot \frac{\partial v_\theta}{\partial \theta} = 0 \right]$$

$$\frac{\partial v_z}{\partial z} = 0 \text{ which means } v_z = v_z (r, t)$$

Invoking $\left[v_r = 0, v_\theta = 0, \dfrac{\partial v_z}{\partial z} = 0, \text{ and } \dfrac{\partial}{\partial \theta} \text{ (any quantity)} = 0 \right]$ in the

Navier-Stokes equations, we finally obtain

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} + v \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial v_z}{\partial r} \right) \tag{8.41}$$

For steady flow, the governing equation becomes

$$\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial v_z}{\partial r} = \frac{1}{\mu} \frac{dp}{dz} \tag{8.42}$$

The boundary conditions are

at $r = 0$, v_z is finite and

at $r = R$, $v_z = 0$.

Equation (8.42) can be written as

$$r \frac{d^2 v_z}{dr^2} + \frac{dv_z}{dr} = \frac{1}{\mu} \cdot \frac{dp}{dz} r$$

or $$\frac{d}{dr} \left(r \frac{dv_z}{dr} \right) = \frac{1}{\mu} \cdot \frac{dp}{dz} r$$

or $$r \frac{dv_z}{dr} = \frac{1}{2\mu} \cdot \frac{dp}{dz} r^2 + A$$

or $$\frac{dv_z}{dr} = \frac{1}{2\mu} \cdot \frac{dp}{dz} r + \frac{A}{r}$$

or $$v_z = \frac{1}{4\mu} \cdot \frac{dp}{dz} r^2 + A \ln r + B$$

at $r = 0$, v_z is finite which means A should be equal to zero and at $r = R$, $v_z = 0$ yields

$$B = -\frac{1}{4\mu} \cdot \frac{dp}{dz} \cdot R^2$$

$$v_z = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left(1 - \frac{r^2}{R^2}\right) \qquad (8.43)$$

This shows that the axial velocity profile in a fully developed laminar pipe flow is having parabolic variation along r.

At $r = 0$, as such, $v_z = v_{z_{\max}}$

$$v_{z_{\max}} = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right) \qquad (8.44a)$$

The average velocity in the channel,

$$v_{z_{av}} = \frac{Q}{\pi R^2} = \frac{\int\limits_0^R 2\pi r v_z(r)\,dr}{\pi R^2}$$

or

$$v_{z_{av}} = \frac{2\pi \dfrac{R^2}{4\mu}\left(-\dfrac{dp}{dz}\right)\left[\dfrac{R^2}{2} - \dfrac{R^4}{4R^2}\right]}{\pi R^2}$$

$$v_{z_{av}} = \frac{R^2}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{1}{2}v_{z_{\max}} \qquad (8.44b)$$

or $\qquad v_{z_{\max}} = 2\,v_{z_{av}} \qquad (8.44c)$

Now, the discharge through a pipe is given by

$$Q = \pi R^2 v_{z_{av}} \qquad (8.45)$$

or $\qquad Q = \pi R^2 \dfrac{R^2}{8\mu}\left(-\dfrac{dp}{dz}\right)$ [From Eq. 8.44b]

or $\qquad Q = -\dfrac{\pi D^4}{128\mu}\left(\dfrac{dp}{dz}\right) \qquad (8.46)$

Equation (8.46) is commonly used in the measurement of viscosity with the help of capillary tube viscometers. Such a viscometer consists of a constant head tank to supply liquid to a capillary tube (Fig. 8.8).
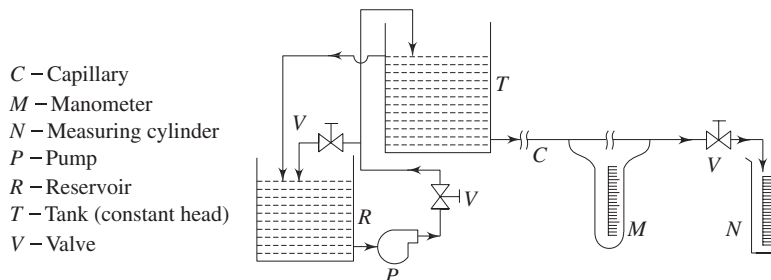
C – Capillary
M – Manometer
N – Measuring cylinder
P – Pump
R – Reservoir
T – Tank (constant head)
V – Valve

Fig. 8.8 Schematic diagram of the experimental facility for determination of viscosity

Pressure drop readings across a specified length in the developed region of the flow are taken with the help of a manometer. The developed flow region is ensured by providing the necessary and sufficient entry length. From Eq. (8.46), the expression for viscosity can be written as

$$\mu = -\frac{\pi D^4}{128\,Q}\cdot\frac{dp}{dl} = \frac{\pi D^4}{128\,Q}\frac{(p_1 - p_2)}{l}$$

The volumetric flow rates (Q) are measured by collecting the liquid in a measuring cylinder. The diameter (D) of the capillary tube is known beforehand. Now the viscosity of a flowing fluid can easily be evaluated.

Shear stress profile across the cross-section can also be determined from this information. Shear stress at any point of the pipe flow given by

$$\tau\,|_r = \mu\,\frac{dv_z}{dr}$$

From Eq. (8.43), $\dfrac{dv_z}{dr} = \dfrac{R^2}{4\mu}\left(\dfrac{dp}{dz}\right)\dfrac{2r}{R^2}$

or
$$\frac{dv_z}{dr} = \frac{1}{2\mu}\left(\frac{dp}{dz}\right)\cdot r \tag{8.47a}$$

which means
$$\tau\,|_r = \frac{1}{2}\left(\frac{dp}{dz}\right)\cdot r \tag{8.47b}$$

This also indicates that τ varies linearly with the radial distance from the axis. At the wall, τ assumes the maximum value.

At $r = R$,
$$\tau = \tau_{\max} = \frac{1}{2}\left(\frac{dp}{dz}\right)R$$

Again, over a pipe length of l, the total shear force is

$$F_s = \tau_{\max}\,2\pi R\cdot l$$

or
$$F_s = \frac{1}{2}\left(\frac{p_2 - p_1}{l}\right)\cdot 2\pi R^2\cdot l$$

or
$$F_s = -\pi R^2 \times [\text{Pressure drop between the specified length}]$$

as it should be. Negative sign indicates that the force is acting in opposite to the flow direction.

However, from Eq. (8.44b), we can write

$$(v_z)_{av} = -\frac{1}{8\mu}\left(\frac{dp}{dz}\right)R^2 \tag{8.47c}$$

or
$$-\left(\frac{dp}{dz}\right) = \frac{8\mu(v_z)_{av}}{R^2} \tag{8.48}$$

Over a finite length l, the head loss $h_f = \dfrac{\text{pressure drop}}{\rho g}$ $\tag{8.49}$

Combining Eqs (8.48) and (8.49), we get

$$h_f = \frac{32\,\mu\,(v_z)_{\text{av}}}{D^2}\cdot l \cdot \frac{1}{\rho g}$$

or
$$h_f = \frac{32\,\mu\,(v_z)^2_{\text{av}}\, l}{D^2\cdot\rho g}\cdot\frac{1}{(v_z)_{\text{av}}} \tag{8.50}$$

On the other hand, the head loss in a pipe flow is given by Darcy-Weisbach formula as

$$h_f = \frac{f\,l\,(v_z)^2_{\text{av}}}{2\,g\,D} \tag{8.51}$$

where "*f*" is Darcy friction factor. Equations (8.50) and (8.51) will yield

$$\frac{32\,\mu\,(v_z)^2_{\text{av}}\,l}{D^2\cdot\rho g}\cdot\frac{1}{(v_z)_{\text{av}}} = \frac{f\,l\,(v_z)^2_{\text{av}}}{2\,g\,D}$$

which finally gives $f = \dfrac{64}{\text{Re}}$, where $\text{Re} = \dfrac{\rho\,(v_z)_{\text{av}}\,D}{\mu}$ is the Reynolds number.

So, for a fully developed laminar flow, the Darcy (or Moody) friction factor is given by

$$f = \frac{64}{\text{Re}} \tag{8.52a}$$

Alternatively, the skin friction coefficient for Hagen-Poiseuille flow can be expressed by

$$C_f = \frac{\left|\tau_{at\ r\,=\,R}\right|}{\dfrac{1}{2}\,\rho\,(v_z)^2_{\text{av}}}$$

With the help of Eqs (8.47b) and (8.47c), it can be written

$$C_f = \frac{16}{\text{Re}} \tag{8.52b}$$

The skin friction coefficient C_f is called as Fanning's friction factor. From comparison of Eqs (8.52a) and (8.52b), it appears

$$f = 4\,C_f$$

For fully developed turbulent flow, the analysis is much more complicated, and we generally depend on experimental results. Friction factor for a wide range of Reynolds number (10^4 to 10^8) can be obtained from a look-up chart which we shall discuss later. Friction factor, for high Reynolds number flows, is also a function of tube surface condition. However, in circular tube, flow is laminar for $\text{Re} \leq 2300$ and turbulent regime starts with $\text{Re} \geq 4000$. In between, transition from laminar to turbulent is induced.

As it has been pointed out, the surface condition of the tube is another responsible parameter in determination of friction factor. Friction factor in the turbulent regime is determined for different degree of surface-roughness $\left(\dfrac{\varepsilon}{D_h}\right)$ of the pipe, where ε is the dimensional roughness and D_h is usually the hydraulic diameter of the pipe. Friction factors for different Reynolds number and surface-roughness have been determined experimentally by various investigators and the comprehensive results are expressed through a graphical presentation which is known as Moody Chart after L.F. Moody who compiled it. This will be presented in detail in Chapter 11.

The hydraulic diameter which is used as the characteristic length in determination of friction factor, instead of ordinary geometrical diameter, is defined as

$$D_h = \frac{4\,A_w}{P_w} \tag{8.53}$$

where A_w is the flow area and P_w is the wetted perimeter.

Now, let us quickly look at the so called kinetic energy correction factor (α) and momentum correction factor (β) associated with fully developed laminar flow.

Kinetic energy correction factor, α   The kinetic energy associated with the fluid flowing with its profile through elemental area $dA = \left[\dfrac{1}{2}\,\rho\,v_z\,dA\right]v_z^2$ and the total

kinetic energy passing through per unit time $= \dfrac{\rho}{2}\displaystyle\int v_z^3 \;dA$. This can be related to

the kinetic energy due to average velocity $(v_z)_{av}$, through a correction factor, α, as

$$\left[\frac{\rho}{2}\,(v_z)_{av}^3\,A\right]\alpha \;=\; \frac{\rho}{2}\int v_z^3 \;dA$$

or
$$\alpha = \frac{1}{A}\int\left[\frac{v_z}{(v_z)_{av}}\right]^3 \;dA$$

Here, for Hagen-Poiseuille flow,

$$\alpha = \frac{1}{A}\int_0^R \left[\frac{(v_z)_{max}\left(1-\left(\dfrac{r}{R}\right)^2\right)}{(v_z)_{max}\cdot\dfrac{1}{2}}\right]^3 2\pi r dr \;=\; 2 \tag{8.54a}$$

Momentum correction factor, β   The momentum associated with fluid flowing with its profile through elemental area $dA = \rho\,v_z^2\,dA$; and the total momentum passing through any particular section per unit time $= \rho\displaystyle\int v_z^2 \;dA$. This can be

related to the momentum rate due to average flow velocity $(v_z)_{av}$, through a correction factor β, as

$$\left[\rho\,(v_z)_{av}^2\,A\right]\beta = \rho \int v_z^2 \; dA \quad \text{or} \quad \beta = \frac{1}{A}\int\left[\frac{v_z}{(v_z)_{av}}\right]^2 \; dA$$

Here, for Hagen-Poiseuille flow,

$$\beta = \frac{1}{A}\int_0^R \left[\frac{(v_z)_{max}\left(1-\left(\frac{r}{R}\right)^2\right)}{(v_z)_{max}\cdot\frac{1}{2}}\right]^2 2\pi\,r\,dr = \frac{4}{3} \tag{8.54b}$$

8.4.4 Flow between Two Concentric Rotating Cylinders

Another example which leads to an exact solution of Navier-Stokes equation is the flow between two concentric rotating cylinders. Consider flow in the annulus of two cylinders (Fig. 8.9), where r_1 and r_2 are the radii of inner and outer cylinders, respectively, and the cylinders move with different rotational speeds ω_1 and ω_2, respectively.
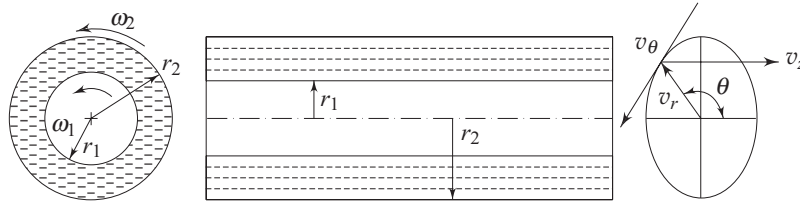

Fig. 8.9 Flow beween two concentric rotating cylinders

From the physics of the problem we know, $v_z = 0$, $v_r = 0$. From the continuity Eq. (8.24) and these two conditions, we obtain

$$\frac{\partial v_\theta}{\partial \theta} = 0$$

which means v_θ is not a function of θ. We assume z dimension to be large enough so that end effects can be neglected and $\dfrac{\partial}{\partial z}$ (any property) $= 0$. Now, we can say $v_\theta = v_\theta(r)$. With these simplifications and assuming that "θ symmetry" holds good, (8.23) reduces to

$$\rho\frac{v_\theta^2}{r} = \frac{dp}{dr} \tag{8.55}$$

and

$$\frac{d^2 v_\theta}{dr^2} + \frac{1}{r}\cdot\frac{dv_\theta}{dr} - \frac{v_\theta}{r^2} = 0 \tag{8.56}$$

Equation (8.55) signifies that the centrifugal force is supplied by the radial pressure, exerted by the wall of the enclosure on the fluid. In other words, it describes the radial pressure distribution.

From Eq. (8.56), we get

$$\frac{d}{dr}\left[\frac{1}{r}\cdot\frac{d}{dr}(rv_\theta)\right] = 0$$

or
$$\frac{d}{dr}(rv_\theta) = Ar \quad \text{or} \quad v_\theta = \frac{Ar}{2} + \frac{B}{r} \tag{8.57}$$

For the azimuthal component of velocity, v_θ, the boundary conditions are: at $r = r_1$, $v_\theta = r_1\,\omega_1$ at $r = r_2$, $v_\theta = r_2\,\omega_2$. Application of these boundary conditions on Eq. (8.57) will produce

$$A = 2\left[\omega_1 - \frac{r_2^2}{(r_2^2 - r_1^2)}(\omega_1 - \omega_2)\right]$$

and
$$B = \frac{r_1^2\,r_2^2}{(r_2^2 - r_1^2)}(\omega_1 - \omega_2)$$

Finally, the velocity distribution is given by

$$v_\theta = \frac{1}{(r_2^2 - r_1^2)}\left[(\omega_2\,r_2^2 - \omega_1\,r_1^2)r + \frac{r_1^2\,r_2^2\,(\omega_1 - \omega_2)}{r}\right] \tag{8.58}$$

Now, $\tau_{r\theta} = \mu\,\dot{\gamma}_{r\theta}$ is the general stress-strain relation.

or
$$\tau_{r\theta} = \mu\left(\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\cdot\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right)$$

In our case,
$$\tau_{r\theta} = \mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

or
$$\tau_{r\theta} = \mu\,r\,\frac{d}{dr}\left(\frac{v_\theta}{r}\right) \tag{8.59}$$

Equations (8.58) and (8.59) yields

$$\tau_{r\theta} = \frac{2\mu}{r_2^2 - r_1^2}(\omega_2 - \omega_1)r_1^2\,r_2^2\,\frac{1}{r^2} \tag{8.60}$$

Now,
$$\tau_{r\theta}\big|_{r=r_1} = \frac{2\mu r_2^2\,(\omega_2 - \omega_1)}{r_2^2 - r_1^2}$$

and
$$\tau_{r\theta}\big|_{r=r_2} = \frac{2\mu r_1^2\,(\omega_2 - \omega_1)}{r_2^2 - r_1^2}$$

For the case, when the inner cylinder is at rest and the outer cylinder rotates, the torque transmitted by the outer cylinder to the fluid is

$$T_2 = \frac{2\mu r_1^2\,\omega_2}{r_2^2 - r_1^2}\cdot 2\pi r_2\,l\,r_2$$

or
$$T_2 = 4\pi\mu l\,\frac{r_1^2\,r_2^2}{r_2^2 - r_1^2}\,\omega_2 \tag{8.61}$$

where l is the length of the cylinder. The moment T_1, with which the fluid acts on the inner cylinder has the same magnitude. If the angular velocity of the external cylinder and the moment acting on the inner cylinder are measured, the coefficient of viscosity can be evaluated by making use of the Eq. (8.61).

8.5 LOW REYNOLDS NUMBER FLOW

We have seen in Chapter 6 that Reynolds number is the ratio of inertia force to viscous force. For flow at low Reynolds number, the inertia terms in the Navier–Stokes equations become small as compared to viscous terms. As such, when the inertia terms are omitted from the equations of motion, the analyses are valid for only Re << 1. Consequently, this approximation, linearizes the Navier–Stokes equations and for some problems, makes it amenable to analytical solutions. We shall discuss such flows in this section. Motions at very low Reynolds number are sometimes referred to as creeping motion.

8.5.1   Theory of Hydrodynamic Lubrication

Thin film of oil, confined between the interspace of moving parts, may acquire high pressures up to 100 MPa which is capable of supporting load and reducing friction. The salient features of this type of motion can be understood from a study of slipper bearing (Fig. 8.10). The slipper moves with a constant velocity U past the bearing plate. This slipper face and the bearing plate are not parallel but slightly inclined at an angle of α. A typical bearing has a gap width of 0.025 mm or less, and the convergence between the walls may be of the order of 1/5000. It is assumed that the sliding surfaces are very large in transverse direction so that the problem can be considered two-dimensional.
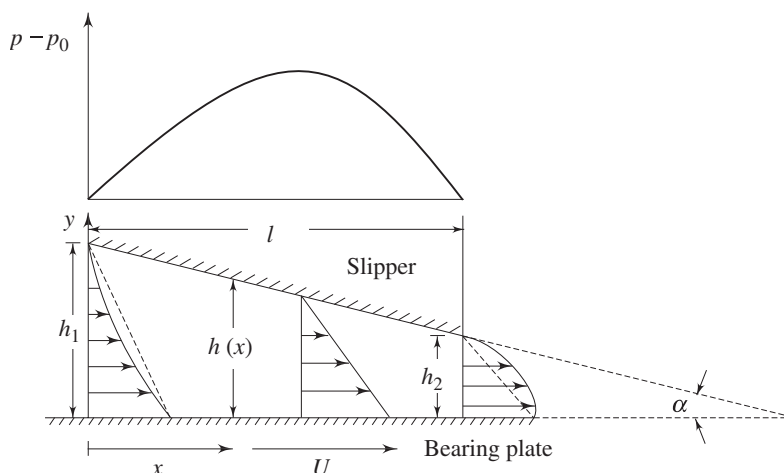

Fig. 8.10   Flow in a slipper bearing

For the analysis, we may assume that the slipper is at rest and the plate is forced to move with a constant velocity U. The height $h(x)$ of the wedge between

the block and the guide is assumed to be very small as compared with the length l of the block. This motion is different from that we have considered while discussing Couette flow. The essential difference lies in the fact that here the two walls are inclined at an angle to each other. Due to the gradual reduction of narrowing passage, the convective acceleration $u \dfrac{\partial u}{\partial x}$ is distinctly not zero. However, a relative estimation of inertia term with respect to viscous term suggests that, for all practical purposes, inertia terms can be neglected. The estimate is done in the following way

$$\frac{\text{Inertia force}}{\text{Viscous force}} = \frac{\rho u(\partial u/\partial x)}{\mu(\partial^2 u/\partial y^2)} = \frac{\rho U^2/l}{\mu U/h^2} = \frac{\rho U l}{\mu}\left(\frac{h}{l}\right)^2$$

The inertia force can be neglected with respect to viscous force if the modified Reynolds number,

$$R^* = \frac{U l}{\nu}\left(\frac{h}{l}\right)^2 \ll 1$$

The equation for motion in y direction can be omitted since the v component of velocity is very small with respect to u. Besides, in the x-momentum equation, $\partial^2 u/\partial x^2$ can be neglected as compared with $\partial^2 u/\partial y^2$ because the former is smaller than the latter by a factor of the order of $(h/l)^2$. With these simplifications the equations of motion reduce to

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx} \tag{8.62}$$

The equation of continuity can be written as

$$Q = \int_0^{h(x)} u \, dy \tag{8.63}$$

The boundary conditions are:
at $y = 0$, $u = U$ at $x = 0$, $p = p_0$
at $y = h$, $u = 0$ and at $x = l$, $p = p_0$ (8.64)

Integrating Eq. (8.62) with respect to y, we obtain

$$u = \frac{1}{2\mu} \cdot \frac{dp}{dx} y^2 + C_1 y + C_2$$

Application of the kinematic boundary conditions (at $y = 0$, $u = U$ and at $y = h$, $u = 0$), yields

$$u = U\left(1 - \frac{y}{h}\right) - \frac{h^2}{2\mu} \cdot \frac{dp}{dx}\left(1 - \frac{y}{h}\right)\frac{y}{h} \tag{8.65}$$

It is to be noticed that $\left(\dfrac{dp}{dx}\right)$ is constant as far as integration along y is concerned, but p and $\dfrac{dp}{dx}$ vary along x-axis. At the point of maximum pressure, $\dfrac{dp}{dx} = 0$, hence

$$u = U \left(1 - \frac{y}{h} \right) \qquad (8.66)$$

Equation (8.66) depicts that the velocity profile along y is linear at the location of maximum pressure. The gap at this location may be denoted as h^*.

Now, substituting Eq. (8.65) into Eq. (8.63) and integrating, we get

$$Q = \frac{Uh}{2} - \frac{p'h^3}{12\,\mu}$$

or $\qquad p' = 12\,\mu \left(\frac{U}{2h^2} - \frac{Q}{h^3} \right) \qquad (8.67)$

where $\qquad p' = \mathrm{d}p/\mathrm{d}x.$

Integrating Eq. (8.67) with respect to x, we obtain

$$\int \frac{\mathrm{d}p}{\mathrm{d}x}\,\mathrm{d}x \;=\; 6\mu U \int \frac{\mathrm{d}x}{(h_1 - \alpha x)^2} - 12\,\mu Q \int \frac{\mathrm{d}x}{(h_1 - \alpha x)^3} + C_3 \qquad (8.68a)$$

or $\qquad p = \dfrac{6\mu U}{\alpha(h_1 - \alpha x)} - \dfrac{6\mu Q}{\alpha(h_1 - \alpha x)^2} + C_3 \qquad (8.68b)$

where $\qquad \alpha = (h_1 - h_2)/l \qquad$ and $\qquad C_3$ is a constant.

Since the pressure must be the same ($p = p_0$), at the ends of the bearing, namely, $p = p_0$ at $x = 0$ and $p = p_0$ at $x = l$, the unknowns in the above equations can be determined by applying the pressure boundary conditions. We obtain

$$Q = \frac{U h_1 h_2}{h_1 + h_2} \qquad \text{and} \qquad C_3 = p_0 - \frac{6\mu U}{\alpha(h_1 + h_2)}$$

With these values inserted, the equation for pressure distribution (8.68) becomes

$$p = p_0 + \frac{6\,\mu\, Ux\,(h - h_2)}{h^2\,(h_1 + h_2)}$$

or $\qquad p - p_0 = \dfrac{6\,\mu\, Ux\,(h - h_2)}{h^2\,(h_1 + h_2)} \qquad (8.69)$

It may be seen from Eq. (8.69) that, if the gap is uniform, i.e. $h = h_1 = h_2$, the gauge pressure will be zero. Furthermore, it can be said that very high pressure can be developed by keeping the film thickness very small. Figure 8.10 shows the distribution of pressure throughout the bearing.

The total load bearing capacity per unit width is

$$P = \int_0^l (p - p_0)\,\mathrm{d}x = \frac{6\mu U}{h_1 + h_2} \int_0^l \frac{x(h - h_2)}{h^2}\,\mathrm{d}x$$

After substituting $h = h_1 - \alpha x$ with $\alpha = (h_1 - h_2)/l$ in the above equation and performing the integration,

$$P = \frac{6\mu U l^2}{(h_1 - h_2)^2} \left[\ln \frac{h_1}{h_2} - 2 \left\{ \frac{h_1 - h_2}{h_1 + h_2} \right\} \right] \qquad (8.70)$$

The shear stress at the bearing plate is

$$\tau_0 = -\mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \left(p'\frac{h}{2} + \mu\frac{U}{h}\right) \tag{8.71}$$

Substituting the value of p' from Eq. (8.67) and then invoking the value of Q in Eq. (8.71), the final expression for shear stress becomes

$$\tau_0 = 4\mu\frac{U}{h} - \frac{6\mu U h_1 h_2}{h^2(h_1 + h_2)}$$

The drag force required to move the lower surface at speed U is expressed by

$$D = \int_0^l \tau_0 \, dx = \frac{\mu U l}{h_1 - h_2}\left[4\ln\frac{h_1}{h_2} - 6\frac{h_1 - h_2}{h_1 + h_2}\right] \tag{8.72}$$

Michell thrust bearing, named after A.G.M. Michell, works on the principles based on the theory of hydrodynamic lubrication. The journal bearing(Fig. 8.11) develops its force by the same action, except that the surfaces are curved.
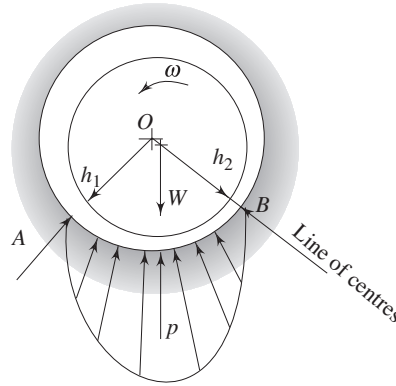

Fig. 8.11 Hydrodynamic action of a journal bearing

8.5.2 Low Reynolds Number Flow Around a Sphere

Stokes obtained the solution for the pressure and velocity field for the slow motion of a viscous fluid past a sphere. In his analysis, Stokes neglected the inertia terms of Navier-Stokes equations. Details of the solutions are beyond the scope of this text. However, integrating the pressure distribution and the shearing stress over the surface of a sphere of radius R, Stokes found that the drag D of the sphere, which is placed in a parallel stream of uniform velocity U_∞, is given by

$$D = 6\,\pi\mu\,RU_\infty \tag{8.73}$$

This is the well-known Stokes' equation for the drag of a sphere. It can be shown that one third of the total drag is due to pressure distribution and the remaining two third arises from frictional forces. If the drag coefficient is defined according to the relation

$$C_D = \frac{D}{\dfrac{1}{2}\rho U_\infty^2 A} \tag{8.74}$$

where $\left(A = \dfrac{\pi}{4} d^2 \right)$ is the frontal area of the sphere, then

$$C_D = \frac{6\pi\mu R U_\infty}{\dfrac{1}{2}\rho U_\infty^2 \dfrac{\pi}{4} d^2}$$

or
$$C_D = \frac{24}{\text{Re}}; \ \text{Re} = \frac{U_\infty d}{\nu} \tag{8.75}$$

A comparison between Stokes' drag coefficient in Eq. (8.75) and experiments is shown in Fig. 8.12. The approximate solution due to Stokes' is valid for Re < 1.
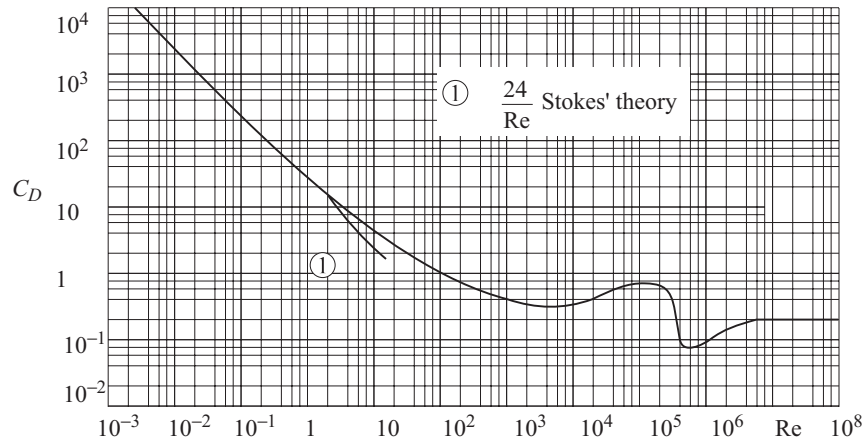

Fig. 8.12 Comparison between Stokes' drag coefficient and experimental drag coefficient

An important application of Stokes' law is the determination of viscosity of a viscous fluid by measuring the terminal velocity of a falling sphere. In this device, a sphere is dropped in a transparent cylinder containing the fluid under test. If the specific weight of the sphere is close to that of the liquid, the sphere will approach a small constant speed after being released in the fluid. Now we can apply Stokes' law for steady creeping flow around a sphere where the drag force on the sphere is given by Eq. (8.73).

With the sphere, falling at a constant speed, the acceleration is zero. This signifies that the falling body has attained terminal velocity and we can say that the sum of the buoyant force and drag force is equal to weight of the body.

$$\frac{4}{3}\pi R^3 \rho_s g = \frac{4}{3}\pi R^3 \rho_l g + 6\pi\mu V_T R \tag{8.76}$$

where ρ_s is the density of the sphere, ρ_l is density of the liquid and V_T is the terminal velocity.

Solving for μ, we get

$$\mu = \frac{2}{9}\frac{gR^2}{V_T}(\rho_s - \rho_l) \tag{8.77}$$

The terminal velocity V_T can be measured by observing the time for the sphere to cross a known distance between two points after its acceleration has ceased.

Summary

- The Navier–Stokes equations, based on the conservation of momentum have been derived for a viscous incompressible fluid.
- The Navier–Stokes equations are not amenable to an analytical solution due to the presence of nonlinear inertia terms in it. However, there are some special situations where the nonlinear inertia terms are reduced to zero. In such situations, exact solutions of the Navier-Stokes equations are obtainable. This includes the plane Poiseuille flow, the Couette flow, the flow through a straight pipe and the flow between two concentric rotating cylinders. All these flows are known as parallel flow where only one component of the velocity is non-trivial.
- Knowledge of the velocity field obtained through analytical methods permits calculation of shear stress, pressure drop and flow rate. Applications of the parallel flow theory to the measurement of viscosity and hydrodynamics of bearing-lubrication are explained.

References

1. White, F.M., *Viscous Fluid Flow*, McGraw-Hill Book Company, 1991.
2. Faber, T.E., *Fluid Dynamics for Physicists*, Combridge University Press, 1995.

Solved Examples

Example 8.1   Two infinite plates are at h distance apart as in Fig. 8.13. There is a fluid of viscosity μ between the plates and the pressure is constant. The upper plate is moving at speed $U = 4$ m/s. The height of the channel $h = 1.8$ cm. Calculate the shear stress at the upper and lower walls if $\mu = 0.44$ kg/m.s and $\rho = 888$ kg/m^3.
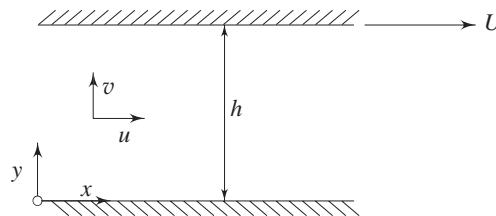

Fig. 8.13 Parallel flow between two plates with upper plate moving

Solution   Re $= \rho h U/\mu = (888) (1.8/100) (4)/0.44 = 145$. So, the flow is laminar and $\tau = \mu \dfrac{\partial u}{\partial y}$, u at any y is given by $\dfrac{U}{h} y$.

Shear stresses at the two walls are of equal magnitude, therefore,

$$\tau = \mu \frac{\partial u}{\partial y} = \mu \frac{(U-0)}{h} \ (0.44) \ (4)/(1.8\,/100)$$

$$= 97.8 \text{ Pa}$$

Example 8.2 Water at 60° flows between two large flat plates. The lower plate moves to the left at a speed of 0.3 m/s. The plate spacing is 3 mm and the flow is laminar. Determine the pressure gradient required to produce zero net flow at a cross-section. ($\mu = 4.7 \times 10^{-4}$ Ns/m^2 at 60 °C)
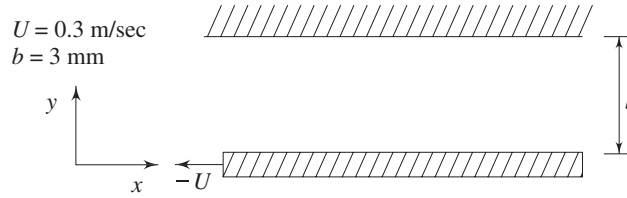

Fig. 8.14

Solution Governing equation: $\mu \dfrac{d^2 u}{dy^2} = \dfrac{dp}{dx}$

$$u = \frac{1}{2\mu} \frac{dp}{dx} y^2 + C_1 y + C_2$$

at $y = 0$, $u = -U$, $\qquad C_2 = -U$
at $y = b$, $u = 0$, which yields

$$0 = \frac{1}{2\mu} \frac{dp}{dx} b^2 + C_1 b - U,$$

or $\qquad C_1 = \dfrac{U}{b} - \dfrac{1}{2\mu} \cdot \dfrac{dp}{dx} b$

$$u = \frac{1}{2\mu} \frac{dp}{dx} (y^2 - by) + U\left(\frac{y}{b} - 1\right)$$

Now, $\qquad Q = \displaystyle\int_0^b u\,dy = \int_0^b \left[\frac{1}{2\mu} \frac{dp}{dx}(y^2 - by) + U\left(\frac{y}{b} - 1\right) \right] dy$

or $\qquad Q = -\dfrac{1}{12\mu} \dfrac{dp}{dx} b^3 - \dfrac{Ub}{2}$

For, $\qquad Q = 0$, with $\mu = 4.7 \times 10^{-4}$ Ns/m^2

$$\frac{dp}{dx} = -\frac{6U\mu}{b^2} = \frac{-6 \times 0.3 \times 4.7 \times 10^{-4}}{(0.003)^2} = -94.0 \text{ N/m}^2 \cdot \text{m}$$

Example 8.3 A continuous belt (Fig. 8.15) passing upward through a chemical bath at velocity U_0, picks up a liquid film of thickness, h, density ρ, and viscosity μ. Gravity tends to make the liquid drain down, but the movement of the belt keep the fluid from running off completely. Assume that the flow is fully developed and that the

atmosphere produces no shear at the outer surface of the film. State clearly the boundary conditions to be satisfied by velocity at $y = 0$ and $y = h$. Obtain an expression for the velocity profile.
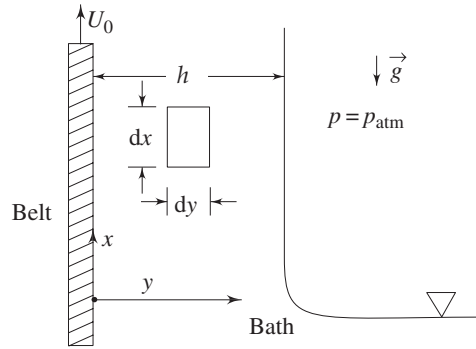

Fig. 8.15

Solution The governing equation is

$$\mu \frac{d^2 u}{dy^2} = \rho g$$

or

$$\mu \frac{du}{dy} = \rho g y + C_1$$

or

$$\frac{du}{dy} = \frac{\rho g y}{\mu} + \frac{C_1}{\mu}$$

$$u = \frac{\rho g y^2}{2\mu} + \frac{C_1}{\mu} y + C_2$$

at $y = 0$, $u = U_0$, so $\quad C_2 = U_0$

at $y = h$, $\tau = 0$, so $\quad \dfrac{du}{dy} = 0$ and $C_1 = -\rho g h$

$$u = \frac{\rho g y^2}{2\mu} - \frac{\rho g h y}{\mu} + U_0 = \frac{\rho g}{\mu}\left(\frac{y^2}{2} - hy\right) + U_0$$

Example 8.4 Water enters a rectangular duct at a rate of $10 \text{ m}^3/\text{s}$ as shown below. Two of the faces of the duct are porous. On the upper face, water is added at a rate shown by the parabolic curve, while on the front face water leaves at a rate determined linearly by the distance from the end. The maximum values of both rates are as given in Fig. 8.16. What is the average velocity leaving the duct if it is 1m long and has a cross-section of 0.1 m^2?
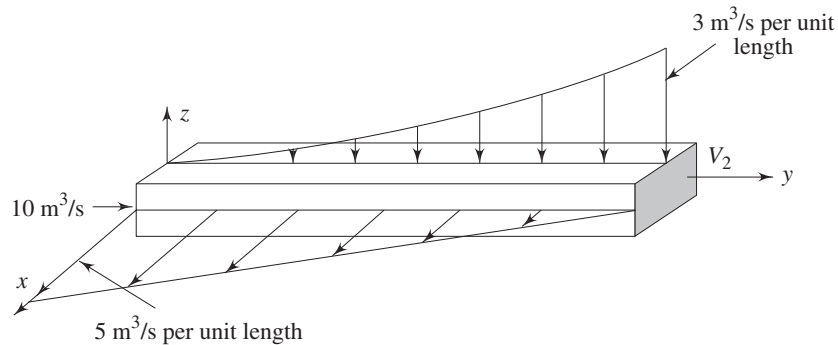
Fig. 8.16

Solution Consider the top surface. The water enters the top surface in a parabolic manner. Let us first find this parabolic curve.

Let $w = ay^2 + by + c$, where w is the flow rate per unit length at top face. Following are the boundary conditions:

at $y = 0$, $w = 0$

at $y = 1$, $w = 3$

at $y = 0$, $\dfrac{dw}{dy} = 0$

So, $c = b = 0$, $a = 3$

Thus, $w = 3y^2$

Similarly, consider the front surface.

let $u = my + d$

when $\left. \begin{array}{l} y = 1, u = 0 \\ y = 0, u = 5 \end{array} \right\}$ we get: $d = 5$

$m = -5$

So, $u = -5y + 5$

For steady incompressible flow, the continuity equation gives

$$\int_{cs} \vec{V} \cdot d\vec{A} = 0$$

Choose the interior of the duct as a control volume.

Thus, $-10 - \displaystyle\int_0^1 3y^2 dy + \int_0^1 (5 - 5y) dy + V_2(0.1) = 0$

or $-10 - 1 + \left(5 - \dfrac{5}{2}\right) = -0.1\, V_2$

Finally, $V_2 = 85$ m/s

Example 8.5 Water at 20 °C is flowing between a two dimensional channel in which the top and bottom walls are 1.5 mm apart. If the average velocity is 2 m/s, find out

(a) the maximum velocity, (b) the pressure drop, (c) the wall shearing stress, and (d) the friction coefficient [$\mu = 0.00101$ kg/m · s].

Solution (a) The maximum velocity is given by

$$U_{\max} = \frac{3}{2} U_{av} = \frac{3}{2}[2] = 3 \text{ m/s}$$

(b) The pressure drop in a two dimensional straight channel is given by

$$\frac{dp}{dx} = \frac{-2\mu U_{\max}}{b^2}, \text{ where } b = \text{ half the channel height}$$

$$= \frac{-2(0.00101)3}{[1.5/(2 \times 1000)]^2}$$

$$= -10773.33 \text{ N/m}^3$$

or $\qquad -10773.33$ N/m^2 per metre

(c) The wall shearing stress in a channel flow is given by

$$\tau_{yx} = -\mu \frac{\partial u}{\partial y} = -b \frac{dp}{dx} = \frac{-1.5}{2 \times 1000} (-10773.33)$$

or $\qquad \tau_{yx} = 8.080$ N/m^2

(d) Re (based on channel height and average velocity)

$$= \frac{\rho U_{av}(2b)}{\mu} = \frac{1000 \times 2 \times (1.5/1000)}{0.00101} \approx 2970$$

which is more than 2300 but the flow is not turbulent as well. Laminar approximation is quite logical. However, $C_f = \dfrac{12}{\text{Re}} = 0.004$.

Example 8.6 The analysis of a fully developed laminar flow through a pipe can alternatively be derived from control volume approach. Derive the expression

$$v_z = \frac{R^2}{4\mu}\left(-\frac{dp}{dz}\right)\left(1 - \frac{r^2}{R^2}\right) \text{ starting from the control volume approach.}$$

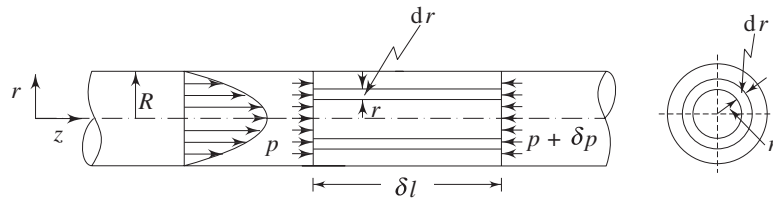**Fig. 8.17** Fully developed laminar flow through a pipe

Solution Let us have a look at Fig. 8.17. The fluid moves due to the pressure gradient which acts in the direction of the axis and in the sections perpendicular to it the pressure may be regarded as constant. Due to viscous friction, individual layers act on each other producing a shearing stress which is proportional to $\dfrac{\partial v_z}{\partial r}$.

In order to establish the condition of equilibrium, we consider a fluid cylinder of length δl and radius r. Now we can write

$$[p - (p + \delta p)]\,\pi\,r^2 = -\,\tau\,2\,\pi\,r\,\delta l$$

or

$$-\,\delta p\,\pi\,r^2 = -\,\mu\,\frac{\partial v_z}{\partial r}\,2\,\pi\,r\,\delta l$$

or

$$\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}l}\,r = \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}z}\,r$$

upon integration,

$$v_z = \frac{1}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}z}\,r^2 + K$$

at $r = R$, $v_z = 0$, hence $K = -\left(\dfrac{1}{4\mu}\dfrac{\mathrm{d}p}{\mathrm{d}z}\right)\cdot R^2$

So, $v_z = \dfrac{R^2}{4\mu}\left(-\dfrac{\mathrm{d}p}{\mathrm{d}z}\right)\left(1 - \dfrac{r^2}{R^2}\right)$

Example 8.7   In the laminar flow of a fluid in a circular pipe, the velocity profile is exactly a parabola. The rate of discharge is then represented by volume of a paraboloid. Prove that for this case the ratio of the maximum velocity to mean velocity is 2.

Solution   See Fig. 8.17. For a paraboloid,

$$v_z = v_{z_{\max}}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$Q = \int v_z\,\mathrm{d}A = \int_0^R v_{z_{\max}}\left[1 - \left(\frac{r}{R}\right)^2\right](2\pi r\,\mathrm{d}r)$$

$$= 2\pi\,v_{z_{\max}}\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = 2\pi\,v_{z_{\max}}\left[\frac{R^2}{2} - \frac{R^2}{4}\right]$$

$$= v_{z_{\max}}\left(\frac{\pi R^2}{2}\right)$$

$$v_{z_{\mathrm{mean}}} = \frac{Q}{A} = \frac{v_{z_{\max}}(\pi R^2/2)}{(\pi R^2)} = \frac{v_{z_{\max}}}{2}$$

Thus, $\dfrac{v_{z_{\max}}}{v_{z_{\mathrm{mean}}}} = 2$

Example 8.8   The velocity distribution for a fully-developed laminar flow in a pipe is given by

$$u = -\frac{R^2}{4\mu} \cdot \frac{\partial p}{\partial z}[1 - (r/R)^2]$$

Determine the radial distance from the pipe axis at which the velocity equals the average velocity.

Solution For a fully-developed laminar flow in a pipe, we can write

$$u = -\frac{R^2}{4\mu}\frac{\partial p}{\partial z}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$V_{av} = \frac{Q}{A} = \frac{1}{\pi R^2}\int_0^R \left\{-\frac{R^2}{4\mu}\frac{\partial p}{\partial z}\left[1 - \left(\frac{r}{R}\right)^2\right]\right\}2\pi r\,dr$$

$$= -\frac{R^2}{8\mu}\frac{\partial p}{\partial z}$$

Now, for $u = V_{av}$ we have,

$$\frac{R^2}{4\mu}\frac{\partial p}{\partial z}\left[1 - \left(\frac{r}{R}\right)^2\right] = -\frac{R^2}{8\mu}\frac{\partial p}{\partial z}$$

or

$$1 - \left(\frac{r}{R}\right)^2 = \frac{1}{2}$$

or

$$\left(\frac{r}{R}\right)^2 = \frac{1}{2} \quad \text{or} \quad r = \frac{R}{\sqrt{2}} = 0.707\,R$$

Example 8.9 SAE 10 oil is flowing through a pipe line at a velocity of 1.0 m/s. The pipe is 45 m long and has a diameter of 150 mm. Find the head loss due to friction. [$\rho = 869$ kg/m^3 and $\mu = 0.0814$ kg/m · s]

Solution $\qquad h_f = \dfrac{f\,l\,V^2}{2gD}$

In order to know f, first we have to calculate Re.

$$\text{Re} = \frac{\rho V D}{\mu}$$

$$= \frac{(869)\,(1)\,(150/1000)}{0.0814} = 1601.35$$

Since Re < 2000, the flow can be assumed to be laminar and

$$f = 64/\text{Re} = 64/1601 = 0.04$$

So, $\qquad h_f = \dfrac{(0.04)(45)(1.0)^2}{(2)(9.81)(150/1000)} = 0.612\text{ m}$

Example 8.10   Heavy fuel oil flows from A to B through a 100 m horizontal steel pipe of 150 mm diameter. The pressure at A is 1.08 MPa and at B is 0.95 MPa. The kinematic viscosity is 412.5×10^{-6} m^2/s, and the relative density of the oil is 0.918. What is the flow rate in m^3/s?

Solution   The Bernoulli's equation between A and B

$$\frac{1.08 \times 10^6}{918 \times 9.81} + \frac{V^2}{2g} + 0 = \frac{0.95 \times 10^6}{918 \times 9.81} + \frac{V^2}{2g} + 0 + f\frac{100\,V^2}{2g \times 0.150}$$

or $\qquad 119.925 - 105.49 = \dfrac{666.67\, f\, V^2}{2g}$

Both V and f are unknown so we have to follow another approach. If laminar flow exists, then from Eq. (8.48),

$$\frac{p_1 - p_2}{l} = \frac{8\mu V_{av}}{R^2}$$

or $\qquad V_{av} = \dfrac{(p_1 - p_2)D^2}{32\mu l} = \dfrac{(1080-950)1000(.150)^2}{32(412.5 \times 10^{-6} \times 918)(100)}$

$$= \frac{130 \times 1000 \times 0.0225}{32 \times 412.5 \times 918 \times 100} \times 10^6 = 2.41 \text{ m/s}$$

$$\text{Re} = \frac{2.41 \times (150/1000)}{412.5 \times 10^{-6}} = 876.36$$

Hence, laminar flow assumption is valid.

$$Q = AV_{av} = \frac{\pi}{4}(0.15)^2 \times 2.41 = 0.0425 \text{ m}^3/\text{s}$$

Example 8.11   A wind tunnel has a wooden ($\varepsilon = 0.0001$ m) rectangular section 40 cm by 1 m by 50 m long. The average velocity is 45 m/s for air at sea-level standard conditions. Find the power required if the fan has 65 percent efficiency. For air, $\rho = 1.2$ kg/m^3, $\mu = 1.81 \times 10^{-5}$ kg/m-s.

Solution   The wetted perimeter of the duct is

$$P_w = \left[\frac{40}{100} + \frac{40}{100} + 1 + 1\right] = 2.8 \text{ m}$$

and the flow area is $A_w = \dfrac{40}{100} \times 1 = 0.4$ m^2.

Hence, hydraulic diameter $D_h = \dfrac{4 \times 0.4}{2.8} = 0.5714$ m

$$\text{Re} = \frac{\rho D_h V_{av}}{\mu} = \frac{(1.20)(0.5714)(45)}{(1.81 \times 10^{-5})} = 1.7 \times 10^6$$

Now, $\qquad \varepsilon/D_h = \dfrac{0.0001}{0.5714} = 0.000175$

From Moody's chart for Re $= 1.7 \times 10^6$ and $\varepsilon/D_h = 0.000175$

$$f = 0.0140$$

$$h_f = f\left(\frac{L}{D_h}\right)\frac{V^2}{2g} = (0.0140)\left(\frac{50}{0.5714}\right)\left[\frac{(45)^2}{2 \times 9.8}\right]$$

$$= 126.5 \, \text{m}$$

and pressure drop, $\Delta p = \rho g h_f = (1.20)(9.8)(126.5) = 1489 \, \text{Pa}$

$$Q = AV = \left(\frac{40}{100}\right)(1)(45) = 18.0 \, \text{m}^3/\text{s}$$

Pumping power, $\quad P = \dfrac{\rho g Q h_f}{\eta} = \dfrac{(1.20)(9.8)(18.0)(126.5)}{0.65}$

$$= 41200 \, \text{W}$$

Example 8.12   A circular pipe of radius a and length L is attached to a smoothly rounded outlet of a liquid reservoir by means of flanges and bolts as shown in Fig. 8.18. At the flange section the velocity is uniform over the cross-section with magnitude V_0. At the outlet, which discharges into the atmosphere, the velocity profile is parabolic because of the friction in the pipe. What force must be supplied by the bolts to hold the pipe in place?
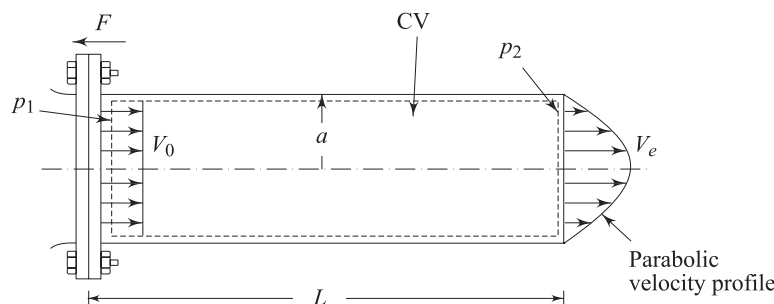

Fig.   8.18

Solution   For the control volume as shown, continuity equation gives (for steady flow)

$$\int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \quad \text{or,} \quad -V_0 \pi a^2 + \int_0^a 2V_e \pi r \, dr = 0, \text{ So, } \int_0^a V_e r \, dr = \frac{V_0 a^2}{2}$$

since V_e is parabolic, let $V_e = a_0 + a_1 r + a_2 r^2$

at $r = a, V_e = 0 \rightarrow a_1 a + a_2 a^2 + a_0 = 0$

at $r = 0, \quad \dfrac{dV_e}{dr} = 0 \rightarrow a_1 = 0$

This will give

$$\int_0^a (a_0 + a_2 r^2) \, r \, dr = \frac{V_0 a^2}{2}$$

or $$a_0 \cdot \frac{a^2}{2} + a_2 \frac{a^4}{4} = V_0 \frac{a^2}{2} \quad \therefore \quad a_0 + \frac{a^2}{2} a_2 = V_0$$

Again, we know $a_0 + a^2 a_2 = 0$

Combining the above two expressions, we get,

$$\frac{a^2}{2} a_2 = -V_0 \quad \therefore \quad a_2 = -\frac{2V_0}{a^2} \quad \text{and} \quad a_0 = + 2V_0$$

$\therefore$ $$V_e = 2V_0 \left[1 - \frac{r^2}{a^2} \right]$$

Now, applying momentum equation to the control volume in the flow direction (let F be the force as shown on the control volume; same force must be supplied by bolts):

$$F + (p_1 - p_2) \pi a^2 = -\rho V_0 \pi a^2 \cdot V_0 + \rho \int_0^a 2\pi r \, dr \, V_e^2$$

$$= -\pi \rho V_0^2 a^2 + \pi \rho V_0^2 \int_0^a 8r \left(1 - \frac{r^2}{a^2} \right)^2 dr$$

$$= \pi \rho V_0^2 a^2 \left[-1 + \int_0^1 8 \frac{r}{a} \left(1 + \frac{r^4}{a^4} - 2\frac{r^2}{a^2} \right) d\left(\frac{r}{a} \right) \right]$$

$$= \pi \rho V_0^2 a^2 \left[-1 + \left(4 + \frac{4}{3} - 4 \right) \right]$$

$$= \frac{1}{3} \pi \rho a^2 V_0^2$$

$\therefore$ $F = \dfrac{1}{3} \pi \rho a^2 V_0^2 - (p_1 - p_2)\pi a^2$ in horizontal direction only (gravity is in vertical direction).

Example 8.13 A slipper (slider) and plate (guide), both 0.5 m wide constitutes a bearing as shown in Fig. 8.19. Density of the fluid, $\rho = 9.00$ kg/m^3 and viscosity, $\mu = 0.1$ Ns/m^2.
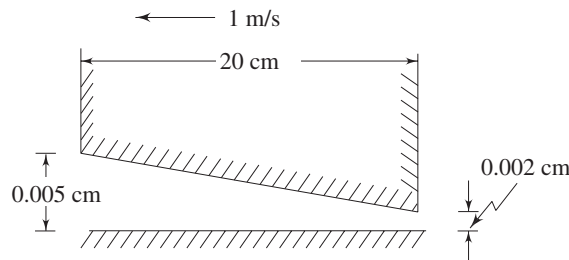

Fig. 8.19   Slipper and plate both 0.5 m wide

Find out the (a) load carrying capacity of the bearing, (b) drag, and (c) power lost in the bearing.

Solution   (a) Considering the width as b and using Eq. (8.70) for the load carrying capacity,

$$P = \frac{6\pi U l^2 b}{(h_1 - h_2)^2}\left[\ln\frac{h_1}{h_2} - 2\frac{h_1 - h_2}{h_1 + h_2}\right]$$

$$= \left[\frac{6 \times 0.1 \times 1 \times 0.2 \times 0.2 \times 0.5}{(0.005 - 0.002)^2}\right] \times$$

$$\left[\ln\frac{0.005}{0.002} - 2\frac{0.005 - 0.002}{0.005 + 0.002}\right]$$

$$= \frac{0.012}{9.0 \times 10^{-6}}\ (0.9163 - 0.8571) = 78.93\ \text{N}$$

(b) Making use of Eq. (8.72) for width b, the drag force may be written as

$$D = \frac{\mu U l b}{h_1 - h_2}\left[4\ln\frac{h_1}{h_2} - 6\frac{h_1 - h_2}{h_1 + h_2}\right]$$

$$= \frac{0.1 \times 1 \times 0.2 \times 0.5}{(0.005 - 0.002)}\left[4\ln\frac{0.005}{0.002} - 6\frac{0.005 - 0.002}{0.005 + 0.002}\right]$$

$$= \frac{0.01}{0.003}\ (3.6651 - 2.5714) = 3.645\ \text{N}$$

(c) Power lost = drag × velocity
$$= 3.645 \times 1 = 3.645\ \text{W}$$

Example 8.14   A cylindrical journal bearing supports a load directed vertically upwards, with the shaft rotating clockwise. Sketch the position of the shaft centre with respect to that of the bearing (hole), if no cavitation is present. Give explanation. No equations are required.

(a)

Solution   In this case, the pressure distribution is symmetric as shown (Fig 8.20) where θ is measured in the direction of the rotation of shaft, from the position of maximum clearance. Thus, as shown, the shaft centre is to the *right* of bearing centre, and the line of centres is *horizontal*.
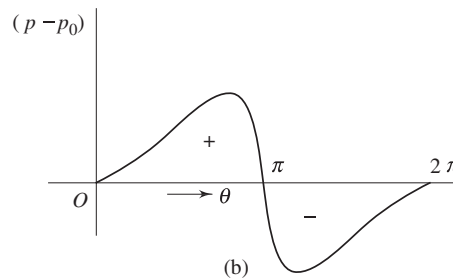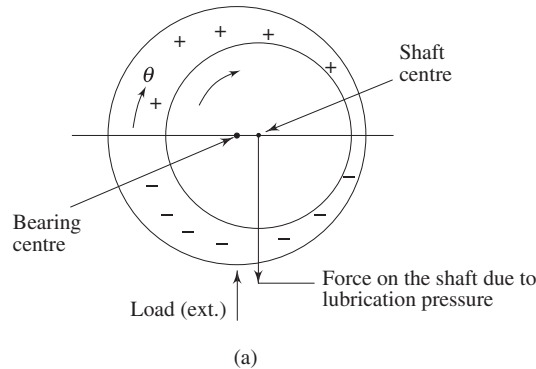
(b)

Fig. 8.20

Example 8.15 For the following thrust bearing (Fig. 8.21), show that the force on the straight slider in the x-direction is the same as that on the guide.
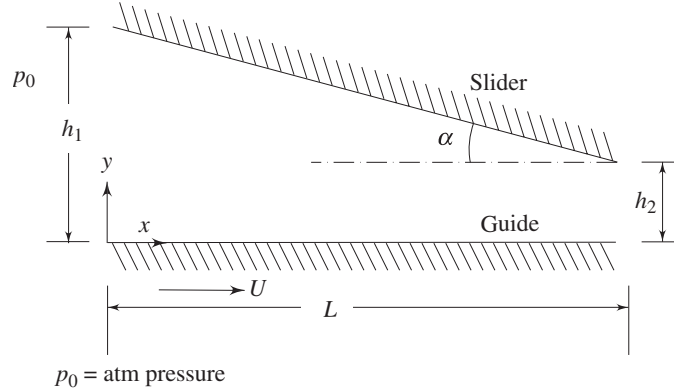
Fig. 8.21

Solution It is given that the velocity profile is

$$\frac{u}{U} = \left(1 - \frac{y}{h}\right)\left[1 - 3\frac{y}{h}\left(1 - \frac{2}{n+1}\frac{h_1}{h}\right)\right]$$

and load

$$P = \frac{6\mu U L^2}{h_2^2(n-1)^2}\left[\ln n - \frac{2(n-1)}{n+1}\right]$$

where $\qquad n = h_1/h_2$

Force on the slider in the x-direction is

$$F_s = \int_0^L \tau_s \, dx(1)\frac{\cos\alpha}{\cos\alpha} + \int_0^L (p - p_0)\frac{dx}{\cos\alpha}(1)\sin\alpha$$

$$= \int_0^L \tau_s \, dx + \tan\alpha \int_0^L (p - p_0)\, dx$$

Now, $\qquad \tau_s = -\mu\left(\frac{\partial u}{\partial y}\right)_{y=h}$

and $\qquad u = U\left(1 - \frac{y}{h}\right)\left[1 - 3\frac{y}{h}\left(1 - \frac{2}{n+1}\cdot\frac{h_1}{h}\right)\right]$, so we get

$$\frac{\partial u}{\partial y} = U\left[-\frac{1}{h} - 3\left(\frac{1}{h} - \frac{2y}{h^2}\right)\left(1 - \frac{2}{n+1}\cdot\frac{h_1}{h}\right)\right]$$

$\therefore \qquad \tau_s = -\mu U\left[-\frac{1}{h} + \frac{3}{h}\left(1 - \frac{2}{n+1}\cdot\frac{h_1}{h}\right)\right]$

$$= \mu U \left[-\frac{2}{h} + \frac{6}{n+1} \cdot \frac{h_1}{h^2} \right]$$

Also,

$$h = h_1 - (h_1 - h_2)\frac{x}{L}$$

$\therefore$

$$dh = -\frac{h_1 - h_2}{L}\, dx$$

Thus,

$$\int_0^L \tau_s\, dx = \frac{L}{h_1 - h_2} \int_{h_2}^{h_1} \tau_s\, dh$$

$$= \frac{\mu L U}{h_1 - h_2} \int_{h_2}^{h_1} \left(-\frac{2}{h} + \frac{6}{n+1} \cdot \frac{h_1}{h^2} \right) dh$$

$$= \frac{\mu U L}{h_1 - h_2} \left(-2 \ln h - \frac{6}{n+1} \cdot \frac{h_1}{h} \right)_{h_2}^{h_1}$$

$$= \frac{\mu U L}{h_2(n-1)} \left[-2 \ln n - \frac{6h_1}{n+1}\left(\frac{1}{h_1} - \frac{1}{h_2} \right) \right]$$

$$= \frac{\mu U L}{h_2(n-1)} \left[-2 \ln n + \frac{6(n-1)}{n+1} \right]$$

Also, load

$$P = \int_0^L (p - p_0) \frac{dx \cos \alpha}{\cos \alpha} \qquad \text{(neglecting contribution of } \tau_s \text{ to load; } \alpha \text{ is small)}$$

$\therefore$

$$F_s = \int_0^L \tau_s\, dx + \tan \alpha\,(P)$$

or

$$F_s = \frac{\mu U L}{h_2(n-1)}\left(-2 \ln n + \frac{6(n-1)}{n+1} \right) + \left[\frac{h_1 - h_2}{L} \right] \times$$

$$\frac{6 \mu U L^2}{h_2^2 (n-1)^2}\left(\ln n - \frac{2(n-1)}{n+1} \right)$$

$$= \frac{\mu U L}{h_2(n-1)}\left(4 \ln n - \frac{6(n-1)}{n+1} \right)$$

Now the force on the guide is

$$F_G = \int_0^L \tau_G\, dx$$

But

$$\tau_G = -\mu \left(\frac{\partial u}{\partial y} \right)_{y=0}$$

$$= \mu U \left[\frac{1}{h} + \frac{3}{h} \left(1 - \frac{2}{n+1} \cdot \frac{h_1}{h} \right) \right]$$

The expression is

$$\tau_G = \mu U \left[\frac{4}{h} - \frac{6}{n+1} \cdot \frac{h_1}{h^2} \right]$$

$\therefore$

$$F_G = \int_0^L \tau_G \, \mathrm{d}x$$

$$= \frac{L}{h_1 - h_2} \int_{h_2}^{h_1} \tau_G \, \mathrm{d}h \qquad \text{(as before)}$$

$$= \frac{\mu U L}{h_2 \, (n-1)} \int_{h_2}^{h_1} \left(\frac{4}{h} - \frac{6}{n+1} \cdot \frac{h_1}{h^2} \right) \mathrm{d}h$$

$$= \frac{\mu U L}{h_2 \, (n-1)} \left[4 \ln n + \frac{6 h_1}{n+1} \left(\frac{1}{h_1} - \frac{1}{h_2} \right) \right]$$

$$= \frac{\mu U L}{h_2 \, (n-1)} \left[4 \ln n - \frac{6(n-1)}{n+1} \right]$$

which is the same as F_s.

Exercises

8.1 Choose the correct answer.
  (i) Bulk stress is equal to thermodynamic pressure
      (a) if second coefficient of viscosity is zero
      (b) for incompressible flows
      (c) for a compressible fluid with negligible second coefficient of viscosity
      (d) if bulk coefficient of viscosity is non-zero.
  (ii) Assumptions made in derivation of Navier-Stokes equations are:
      (a) continuum, incompressible flow Newtonian fluid and μ = constant
      (b) steady flow, incompressible flow, irrotational flow
      (c) continuum, non-Newtonian fluid, incompressible flow
      (d) continuum, Newtonian fluid, Stokes' hypothesis and isotropy.
  (iii) In a fully developed pipe flow
      (a) pressure gradient is greater than the wall shear stress
      (b) inertia force balances the wall shear stress
      (c) pressure gradient balances the wall shear stress only and has a constant value
      (d) none of the above
  (iv) In the case of fully developed flow through tubes,
      (a) Darcy's friction factor is four times the skin friction coefficient
      (b) Darcy's friction factor and skin friction coefficients are same
      (c) Darcy's friction factor is double the skin friction coefficient

    (d) the skin friction coefficient is greater than the Darcy's friction factor.

  (v) Based on hydrodynamic theory of lubrication, state which of the following are correct.

    (a) The load bearing capacity remains unchanged so long either the slipper or the bearing moves in the same direction while the other is held fixed.

    (b) Reversing the direction of the movement of the slipper, bearing remaining fixed, does not cause any change in load bearing capacity.

    (c) $u\dfrac{\partial u}{\partial x} >> \mu\dfrac{\partial^2 u}{\partial y^2}$

    (d) For a large film thickness, $h(x)$, the maximum pressure location shifts from the middle.

  (vi) Observation on a spherical object falling in a liquid pool is the method of measuring viscosity by making use of Stokes' viscosity law. The falling body attains terminal velocity if

    (a) the weight of the falling body is more than the sum of the buoyancy force and the drag force

    (b) the drag force is equal to the buoyancy force

    (c) the buoyancy force is more than the drag force

    (d) the sum of the buoyancy force and the drag force is equal to the weight of the body.

8.2 (a) What is the basic difference between the Euler's equations of motion and the Navier-Stokes equations?

    (b) In case of flow through a straight tube of circular cross-section with rotational symmetry, the axial component of velocity is the only non-trivial component and all the fluid particles move in the same direction only. Find out the average velocity and the maximum velocity within the tube. If Darcy–Weisbach equation for pressure drop over a finite length is given by $h_f = f\,(L/D)\,(V^2/2g)$, prove that $f = 64/\mathrm{Re}$, where L is the length and D is the diameter of the tube.

8.3 What is the relationship between the average velocity and maximum velocity in case of parallel flow between two fixed parallel plates? What do you understand by inlet region and developed region?

<div align="right">Ans.   ($U_{\max} = 1.5\,U_{\mathrm{av}}$)</div>

8.4 Show that in case of a Couette flow, the shear stress at the horizontal mid-plane of the channel is independent of the pressure gradient imposed on the flow.

8.5 (a) Find out the total load and the frictional resistance on a block moving with a velocity U over a horizontal plate separated by a thin layer of lubricating oil, the thickness of layer being h_1 and h_2 at the edges of the block which has a straight bottom.

    (b) Also show that the volume flow rate of lubricant is given by

$$Q = U\,\frac{h_1\,h_2}{h_1 + h_2}$$

8.6 Oil flows between two parallel plates, one of which is at rest and the other moves with a velocity U, (a) If the pressure is decreasing in the direction of flow at the rate of 5 Pa/m, the dynamic viscosity is 0.05 kg/ms, the spacing of the horizontal plate is 0.04 m and the volumetric flow Q per unit width is 0.02 m^2/s, what is the

velocity U? (b) Calculate U if the pressure is increasing at a rate of 5 Pa/m in the direction of flow.

Ans. (a) 0.97 m/s (b) = 1.027 m/s

8.7 Water flows between two very large, horizontal, parallel flat plates 20 mm apart. If the average velocity of water is 0.15 m/s, what is the shear stress (a) at the lower plate, and (b) 5 mm and 10 mm above the lower plate? Assume $\mu = 1.1 \times 10^{-3}$ Ns/m^2.

Ans. (a) 0.0495 N/m^2 (b) 0.0248 N/m^2

8.8 A Newtonian liquid flows slowly under gravity along an inclined flat surface that makes an angle θ with the horizontal plane. The film thickness is T and it is constant. The flow is two-dimensional. (a) Show that the fluid velocity u along x (flow) direction is given by

$$u = \frac{g \sin \theta}{v} y \left(T - \frac{y}{2} \right)$$

(b) Calculate the average velocity u_{av}, and the volumetric flow rate Q per unit width of the surface. The pressure within the fluid is a function of y alone, where y is the normal to the flow direction. The v-component of velocity is trivial.

Ans. ($U_{av} = g \sin \theta\, T^2/3v$, $Q = g \sin \theta\, T^3/3v$)

8.9 A horizontal circular pipe of outer radius R_1 is placed concentrically inside another circular pipe of inner radius R_2. Considering fully developed laminar flow in the annular space between pipes show that the maximum velocity occurs at a radius R_0 given by

$$R_0 = \left[\frac{R_2^2 - R_1^2}{2 \ln (R_2/R_1)} \right]^{1/2}$$

8.10 The Reynolds number for flow of oil through a 5 cm diameter pipe is 1700. The kinematic viscosity, $v = 1.02 \times 10^{-6}$ m^2/s. What is the velocity at a point 0.625 cm away from the wall.

Ans. (0.03 m/s)

8.11 The velocity along the centre line of the Hagen-Poiseuille flow in a 0.1 m diameter pipe is 2 m/s. If the viscosity of the fluid is 0.07 kg/ms and its specific gravity is 0.92, calculate (a) the volumetric flow rate, (b) shear stress of the fluid at the pipe wall, (c) local skin friction coefficient, and (d) the Darcy friction coefficient.

Ans. (a) 7.854×10^{-3} m^3/s (b) 5.6 N/m^2, (c) 0.012 (d) 0.048

8.12 Kerosene at 10 °C flows steadily at 20 l/min through a 150 m long horizontal length of 5.5 cm diameter cast iron pipe. Compare the pressure drop of the kerosene flow with that of the same flow rate of benzene at 10 °C through the same pipe. For kerosene at 10 °C, $\rho = 820$ kg/m^3 and $\mu = 0.0025$ Ns/m^2 and for benzene $\rho = 899$ kg/m^3 and $\mu = 0.0008$ Ns/m^2. Why do you obtain greater pressure drop for benzene?

8.13 A viscous oil flows steadily between parallel plates. The fully developed velocity profile is given by

$$u = - \frac{h^2}{8\mu} \left(\frac{\partial p}{\partial x} \right) \left[1 - \left(\frac{2y}{h} \right)^2 \right]$$

where the total gap between the plates is $h = 3$ mm and y is the distance from the centre line. The viscosity of the oil is 0.5 Ns/m^2 and the pressure gradient is -1200 N/m^2/m. Find the magnitude and direction of the shear stress on the upper plate, and the volumetric flow rate per metre width of the channel.

Ans.   (a) -1.80 N/m^2, (b) 5.40×10^{-6} m^3/s m

8.14 A fully developed laminar flow is taking place in the annulus between two concentric pipes. The inner pipe is stationary, and the outer pipe is moving in the axial direction with a velocity V_0. Assume the axial pressure gradient to be zero ($dp/dz = 0$). Find out a general expression for the shear stress as a function of radial coordinate. Also find out a general expression for the velocity profile $V_z(r)$.

Ans. (a) $\tau = A/r$, (b) $V_z = V_0 \dfrac{\ln (r/r_i)}{\ln (r_o/r_i)}$

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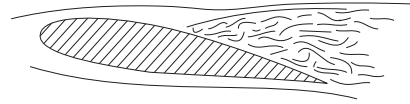
Laminar Boundary Layers

9.1 INTRODUCTION

The boundary layer of a flowing fluid is the thin layer close to the wall. In a flow field, viscous stresses are very prominent within this layer. Although the layer is thin, it is very important to know the details of flow within it. The main-flow velocity within this layer tends to zero while approaching the wall. Also the gradient of this velocity component in a direction normal to the surface is large as compared to the gradient of this component in the streamwise direction.

9.2 BOUNDARY LAYER EQUATIONS

In 1904, Ludwig Prandtl, the well known German scientist, introduced the concept of boundary layer [1] and derived the equations for boundary layer flow by correct reduction of Navier-Stokes equations. He hypothesized that for fluids having relatively small viscosity, the effect of internal friction in the fluid is significant only in a narrow region surrounding solid boundaries or bodies over which the fluid flows. Thus, close to the body is the boundary layer where shear stresses exert an increasingly larger effect on the fluid as one moves from free stream towards the solid boundary. However, outside the boundary layer where the effect of the shear stresses on the flow is small compared to values inside the boundary layer (since the velocity gradient $\partial u/\partial y$ is negligible), the fluid particles experience no vorticity, and therefore, the flow is similar to a potential flow. Hence, the *surface* at the boundary layer interface is a rather fictitious one dividing rotational and irrotational flow. Prandtl's model regarding the boundary layer flow is shown in Fig. 9.1. Hence with the exception of the immediate vicinity of the surface, the flow is frictionless (inviscid) and the velocity is U. In the region, very near to the surface (in the thin layer), there is friction in the flow

which signifies that the fluid is retarded until it adheres to the surface. The transition of the mainstream velocity from zero at the surface to full magnitude takes place across the boundary layer. Its thickness is δ which is a function of the coordinate direction x. The thickness is considered to be very small compared to the characteristic length L of the domain. In the normal direction, within the thin layer, the gradient $\partial u/\partial y$ is very large compared to the gradient in the flow direction $\partial u/\partial x$. Next step is to simplify the Navier–Stokes equations for steady two dimensional laminar incompressible flows. Considering the Navier-Stokes equations together with the equation of continuity, the following dimensional form is obtained.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \tag{9.1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \tag{9.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9.3}$$

Fig. 9.1 Boundary layer on a flat plate

Here the velocity components u and v are acting along the streamwise x and normal y directions respectively. The static pressure is p, while ρ is the density and μ is the dynamic viscosity of the fluid.

The equations are now non-dimensionalised. The length and the velocity scales are chosen as L and U_∞ respectively. The non-dimensional variables are:

$$u^* = \frac{u}{U_\infty},\, v^* = \frac{v}{U_\infty},\, p^* = \frac{p}{\rho U_\infty^2}$$

$$x^* = \frac{x}{L},\, y^* = \frac{y}{L}$$

where U_∞ is the dimensional free stream velocity and the pressure is non-dimensionalised by twice the dynamic pressure $p_d = (1/2)\,\rho\,U_\infty^2$. Using these nondimensional variables, the Eqs (9.1) to (9.3) become

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\text{Re}}\left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right] \tag{9.4}$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{\text{Re}}\left[\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right] \tag{9.5}$$

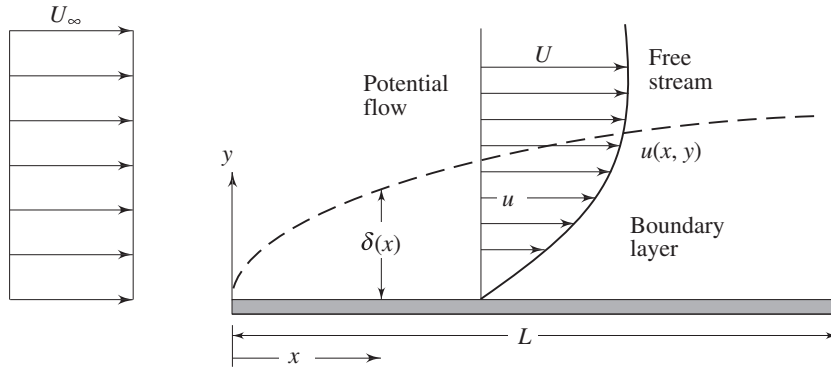$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{9.6}$$

where the Reynolds number,

$$\text{Re} = \frac{\rho U_\infty L}{\mu}$$

Let us examine what happens to the u velocity as we go across the boundary layer. At the wall the u velocity is zero. The value of u on the inviscid side, that is on the free stream side beyond the boundary layer is U. For the case of external flow over a flat plate, this U is equal to U_∞.

Based on the above, we can identify the following scales for the boundary layer variables:

Variable	Dimensional scale	Nondimensional scale
u	U_∞	1
x	L	1
y	δ	$\varepsilon\,(=\delta/L)$

The symbol ε describes a value much smaller than 1. Now, let us look at the order of magnitude of each individual terms involved in Eqs (9.4), (9.5) and (9.6). We start with the continuity Eq. (9.6). One general rule of incompressible fluid mechanics is that we are not allowed to drop any term from the continuity equation. From the scales of boundary layer variables, the derivative $\partial u^*/\partial x^*$ is of the order 1. The second term in the continuity equation $\partial v^*/\partial y^*$ should also be of the order 1. Now, what makes $\partial v^*/\partial y^*$ to have the order 1? Admittedly v^* has to be of the order ε because y^* becomes $\varepsilon\,(=\delta/L)$ at its maximum. Next, consider Eq. (9.4). Inertia terms are of the order 1. Among the second order derivatives, $\partial^2 u^*/\partial x^{*2}$ is of the order 1 and $\partial^2 u^*/\partial y^{*2}$ contains a large estimate of $(1/\varepsilon^2)$. However after multiplication with 1/Re, the sum of these two second order derivatives should produce at least one term which is of the same order of magnitude as the inertia terms. This is possible only if the Reynolds number (Re) is of the order of $1/\varepsilon^2$. It follows from the Eq. (9.4) that $-\partial p^*/\partial x^*$ will not exceed the order of 1 so as to be in balance with the remaining terms. Finally, Eqs (9.4), (9.5) and (9.6) can be rewritten as

$$u^* \frac{\partial u^*}{\partial x^*} \quad + \quad v^* \frac{\partial u^*}{\partial y} \quad = \quad -\frac{\partial p^*}{\partial x^*} \quad + \quad \frac{1}{\text{Re}}\left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right] \tag{9.4}$$

$$(1)\frac{(1)}{(1)} \qquad (\varepsilon)\frac{(1)}{(\varepsilon)} \quad = \quad (1) \qquad (\varepsilon^2)\left[\frac{(1)}{(1)} + \frac{1}{(\varepsilon^2)}\right]$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = - \frac{\partial p^*}{\partial y^*} + \frac{1}{\text{Re}} \left[\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} \right] \tag{9.5}$$

$$(1) \frac{(\varepsilon)}{(1)} \qquad (\varepsilon) \frac{(\varepsilon)}{(\varepsilon)} = (?) \qquad (\varepsilon^2) \left[\frac{(\varepsilon)}{(1)} + \frac{\varepsilon}{(\varepsilon^2)} \right]$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{9.6}$$

$$\frac{(1)}{(1)} \qquad \frac{(\varepsilon)}{(\varepsilon)}$$

As a consequence of the order of magnitude analysis, $\frac{\partial^2 u^*}{\partial x^{*2}}$ can be dropped from the x direction momentum equation, because on multiplication with 1/Re it assumes the smallest order of magnitude. Now, consider the y direction momentum Eq. (9.5). All the terms of this equation are of a smaller magnitude than those of Eq. (9.4). This equation can only be balanced if $\partial p^*/\partial y^*$ is of the same order of magnitude as other terms. Thus the y momentum equation reduces to

$$\frac{\partial p^*}{\partial y^*} = O\,(\varepsilon) \tag{9.7}$$

This means that the pressure across the boundary layer does not change. The pressure is impressed on the boundary layer, and its value is determined by hydrodynamic considerations. This also implies that the pressure p is only a function of x. The pressure forces on a body are solely determined by the inviscid flow outside the boundary layer. The application of Eq. (9.4) at the outer edge of boundary layer gives

$$u^* \frac{\mathrm{d}u^*}{\mathrm{d}x^*} = - \frac{\mathrm{d}p^*}{\mathrm{d}x^*} \tag{9.8a}$$

In dimensional form, this can be written as

$$U \frac{\mathrm{d}U}{\mathrm{d}x} = - \frac{1}{\rho} \frac{\mathrm{d}p}{\mathrm{d}x} \tag{9.8b}$$

On integrating Eq. (9.8b), the well known Bernoulli's equation is obtained,

$$p + \frac{1}{2} \rho U^2 = \text{a constant} \tag{9.9}$$

Finally, it can be said that by the order of magnitude analysis, the Navier-Stokes equations are simplified into equations given below.

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = - \frac{\partial p^*}{\partial x^*} + \frac{1}{\text{Re}} \frac{\partial^2 u^*}{\partial y^{*2}} \tag{9.10}$$

$$\frac{\partial p^*}{\partial y^*} = 0 \tag{9.11}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{9.12}$$

These are known as Prandtl's boundary-layer equations. The available boundary conditions are:

Solid surface

$$\text{at } y^* = 0, \quad u^* = 0 = v^*$$

or $$\text{at } y = 0 \quad u = 0 = v \tag{9.13}$$

Outer edge of boundary-layer

$$\text{at } y^* = (\varepsilon) = \tfrac{\delta}{L}, \quad u^* = 1$$

or $$\text{at } y = \delta, \, u = U(x) \tag{9.14}$$

The unknown pressure p in the x-momentum equation can be determined from Bernoulli's Eq. (9.9), if the inviscid velocity distribution $U(x)$ is also known. The preceding derivations are related to a flat surface, but these can be easily extended to curved surfaces. While doing so, it is seen that Eqs (9.10) to (9.14) continue to be applicable only if the curvature does not change abruptly. However, the boundary layer equations are relatively easier to solve as compared to the Navier-Stokes equations and have been solved by various analytical and numerical techniques.

We solve the Prandtl boundary layer equations for $u^*(x, y)$ and $v^*(x, y)$ with U obtained from the outer inviscid flow analysis. The equations are solved by commencing at the leading edge of the body and moving downstream to the desired location. Note that the reduced momentum Eq. (9.10) is still nonlinear. However, it does allow the no-slip boundary condition to be satisfied which constitutes a significant improvement over the potential flow analysis while solving real fluid flow problems. The *Prandtl boundary layer equations* are thus a simplification of the Navier-Stokes equations.

9.3 BLASIUS FLOW OVER A FLAT PLATE

The classical problem considered by H. Blasius was a two-dimensional, steady, incompressible flow over a flat plate at zero angle of incidence with respect to the uniform stream of velocity U_∞. The fluid extends to infinity in all directions from the plate. The physical problem is already illustrated in Fig. 9.1.

Blasius wanted to determine (a) the velocity field solely within the boundary layer, (b) the boundary layer thickness (δ), (c) the shear stress distribution on the plate, and (d) the drag force on the plate.

The Prandtl boundary layer equations in the case under consideration are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \tag{9.15}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (9.3)$$

The boundary conditions are

$$\begin{aligned}
\text{at } y = 0, \quad & u = v = 0 \\
\text{at } y = \infty \quad & u = U_\infty
\end{aligned} \tag{9.16}$$

It may be mentioned that the substitution of the term $\left[-\dfrac{1}{\rho}\dfrac{\mathrm{d}p}{\mathrm{d}x} \right]$ in the original boundary layer momentum equation in terms of the free stream velocity produces $\left[U_\infty \dfrac{\mathrm{d}U_\infty}{\mathrm{d}x} \right]$ which is equal to zero. Hence the governing Eq. (9.15) does not contain any pressure-gradient term. However, the characteristic parameters of this problem are U_∞, v, x, y, that is,

$$u = u\,(U_\infty,\ v,\ x,\ y)$$

Before we write down this relationship in terms of two non-dimensional parameters, we have to be acquainted with the *law of similarity* in boundary layer flows. It states that the u component of velocity with two velocity profiles of $u(x, y)$ at different x locations differ only by scale factors in u and y. Therefore, the velocity profiles $u(x, y)$ at all values of x can be made congruent if they are plotted in coordinates which have been made dimensionless with reference to the scale factors. The local free stream velocity $U(x)$ at section x is an obvious scale factor for u, because the dimensionless $u(x)$ varies between zero and unity with y at all sections. The scale factor for y, denoted by $g(x)$, is proportional to the local boundary layer thickness so that y itself varies between zero and unity. The principle of similarity demands that the velocity at two arbitrary x locations, namely, x_1 and x_2 should satisfy the equation

$$\frac{u\left[x_1, \{y/g(x_1)\} \right]}{U(x_1)} = \frac{u\left[x_2, \{y/g(x_2)\} \right]}{U(x_2)} \tag{9.17}$$

Now, for Blasius flow, it is possible to write

$$\frac{u}{U_\infty} = F\left(\frac{y}{\sqrt{\dfrac{v\,x}{U_\infty}}} \right) = F(\eta) \tag{9.18}$$

where $\quad \eta \sim \dfrac{y}{\delta}$ and $\delta \sim \sqrt{\dfrac{v\,x}{U_\infty}}$

or more precisely, $\quad \eta = \dfrac{y}{\sqrt{\dfrac{v\,x}{U_\infty}}} \tag{9.19}$

The stream function can now be obtained in terms of the velocity components as

$$\psi = \int u\,\mathrm{d}y = \int U_\infty\,F(\eta)\sqrt{\frac{v\,x}{U_\infty}}\,\mathrm{d}\eta = \sqrt{U_\infty\,v\,x}\int F(\eta)\,\mathrm{d}\eta$$

or $\qquad\qquad \psi = \sqrt{U_\infty\,v\,x}\ f(\eta) + \text{constant} \tag{9.20}$

where $\int F(\eta)\,\mathrm{d}\eta = f(\eta)$ and the constant of integration is zero if the stream function at the solid surface is set equal to zero.

Now, the velocity components and their derivatives are:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = U_\infty \, f'(\eta) \tag{9.21a}$$

$$v = -\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \right) = -\sqrt{U_\infty \nu} \left[\frac{1}{2} \cdot \frac{1}{\sqrt{x}} \, f(\eta) + \sqrt{x} f'(\eta) \left\{ -\frac{1}{2} \cdot \frac{y}{\sqrt{\dfrac{\nu x}{U_\infty}}} \cdot \frac{1}{x} \right\} \right]$$

or
$$v = \frac{1}{2} \sqrt{\frac{\nu U_\infty}{x}} \, [\eta f'(\eta) - f(\eta)] \tag{9.21b}$$

$$\frac{\partial u}{\partial x} = U_\infty f''(\eta) \frac{\partial \eta}{\partial x} = U_\infty f''(\eta) \cdot \left[-\frac{1}{2} \cdot \frac{y}{\sqrt{\dfrac{\nu x}{U_\infty}}} \cdot \frac{1}{x} \right]$$

or
$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2} \cdot \frac{\eta}{x} \cdot f''(\eta) \tag{9.21c}$$

$$\frac{\partial u}{\partial y} = U_\infty f''(\eta) \cdot \frac{\partial \eta}{\partial y} = U_\infty f''(\eta) \cdot \left[\frac{1}{\sqrt{\dfrac{\nu x}{U_\infty}}} \right]$$

or
$$\frac{\partial u}{\partial y} = U_\infty \sqrt{\frac{U_\infty}{\nu x}} \, f''(\eta) \tag{9.21d}$$

$$\frac{\partial^2 u}{\partial y^2} = U_\infty \sqrt{\frac{U_\infty}{\nu x}} \, f'''(\eta) \left\{ \frac{1}{\sqrt{\dfrac{\nu x}{U_\infty}}} \right\}$$

or
$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{\nu x} \, f'''(\eta) \tag{9.21e}$$

Substituting (9.21) into (9.15), we have

$$-\frac{U_\infty^2}{2} \frac{\eta}{x} \cdot f'(\eta) \, f''(\eta) + \frac{U_\infty^2}{2x} [\eta f'(\eta) - f(\eta)] \, f''(\eta) = \frac{U_\infty^2}{x} \, f'''(\eta)$$

or
$$-\frac{1}{2} \frac{U_\infty^2}{x} \, f(\eta) \, f''(\eta) = \frac{U_\infty^2}{x} \, f'''(\eta)$$

or
$$2f'''(\eta) + f(\eta) f''(\eta) = 0 \tag{9.22}$$

This is known as Blasius Equation. The boundary conditions as in Eq. (9.16), in combination with Eq. (9.21a) and (9.21b) become

$$\left. \begin{array}{l} \text{at } \eta = 0: \quad f(\eta) = 0, \quad f'(\eta) = 0 \\ \text{at } \eta = \infty: \quad f'(\eta) = 1 \end{array} \right\} \tag{9.23}$$

Equation (9.22) is a third order nonlinear differential equation. Blasius obtained the solution of this equation in the form of series expansion through analytical techniques which is beyond the scope of this text. However, we shall discuss a numerical technique to solve the aforesaid equation which can be understood rather easily.

It is to be observed that the equation for f does not contain x. Further boundary conditions at $x = 0$ and $y = \infty$ merge into the condition $\eta \to \infty, u/U_\infty = f' = 1$. This is the key feature of similarity solution.

We can rewrite Eq. (9.22) as three first order differential equations in the following way

$$f' = G \tag{9.24a}$$
$$G' = H \tag{9.24b}$$
$$H' = -\frac{1}{2} fH \tag{9.24c}$$

Let us next consider the boundary conditions. The condition $f(0) = 0$ remains valid. Next the condition $f'(0) = 0$ means that $G(0) = 0$. Finally $f'(\infty) = 1$ gives us $G(\infty) = 1$. Note that the equations for f and G have initial values. However, the value for $H(0)$ is not known. Hence, we do not have a usual initial-value problem. Nevertheless, we handle this problem as an initial-value problem by choosing values of $H(0)$ and solving by numerical methods $f(\eta)$, $G(\eta)$, and $H(\eta)$. In general, the condition $G(\infty) = 1$ will not be satisfied for the function G arising from the numerical solution. We then choose other initial values of H so that eventually we find an $H(0)$ which results in $G(\infty) = 1$. This method is called the *shooting technique*.

In Eq. (9.24), the primes refer to differentiation wrt the similarity variable η. The integration steps following Runge-Kutta method are given below

$$f_{n+1} = f_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \tag{9.25a}$$

$$G_{n+1} = G_n + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4) \tag{9.25b}$$

$$H_{n+1} = H_n + \frac{1}{6} (m_1 + 2m_2 + 2m_3 + m_4) \tag{9.25c}$$

One moves from η_n to $\eta_{n+1} = \eta_n + h$. A fourth order accuracy is preserved if h is constant along the integration path, that is, $\eta_{n+1} - \eta_n = h$ for all values of n. The values of k, l and m are as follows.

For generality let the system of governing equations be

$$f' = F_1 (f, G, H, \eta), \quad G' = F_2(f, G, H, \eta) \text{ and } H' = F_3 (f, G, H, \eta).$$

Then,

$$k_1 = h F_1 (f_n, G_n, H_n, \eta_n)$$
$$l_1 = h F_2 (f_n, G_n, H_n, \eta_n)$$
$$m_1 = h F_3 (f_n, G_n, H_n, \eta_n)$$

$$k_2 = h \, F_1 \left\{ \left(f_n + \frac{1}{2} k_1 \right), \left(G_n + \frac{1}{2} l_1 \right), \left(H_n + \frac{1}{2} m_1 \right), \left(\eta_n + \frac{h}{2} \right) \right\}$$

$$l_2 = h \, F_2 \left\{ \left(f_n + \frac{1}{2} k_1 \right), \left(G_n + \frac{1}{2} l_1 \right), \left(H_n + \frac{1}{2} m_1 \right), \left(\eta_n + \frac{h}{2} \right) \right\}$$

$$m_2 = h \, F_3 \left\{ \left(f_n + \frac{1}{2} k_1 \right), \left(G_n + \frac{1}{2} l_1 \right), \left(H_n + \frac{1}{2} m_1 \right), \left(\eta_n + \frac{h}{2} \right) \right\}$$

In a similar way k_3, l_3, m_3 and k_4, l_4, m_4 are calculated following standard formulae for the Runge Kutta integration. For example, k_3 is given by

$$k_3 = hF_1 \left\{ \left(f_n + \frac{1}{2} k_2 \right), \left(G_n + \frac{1}{2} l_2 \right), \left(H_n + \frac{1}{2} m_2 \right), \left(\eta_n + \frac{h}{2} \right) \right\}$$

The functions F_1, F_2 and F_3 are G, H, $-fH/2$ respectively. Then at a distance $\Delta \eta$ from the wall, we have

$$f(\Delta \eta) = f(0) + G(0) \, \Delta \eta \tag{9.26a}$$

$$G(\Delta \eta) = G(0) + H(0) \, \Delta \eta \tag{9.26b}$$

$$H(\Delta \eta) = H(0) + H'(0) \, \Delta \eta \tag{9.26c}$$

$$H'(\Delta \eta) = -\frac{1}{2} \, f(\Delta \eta) \, H(\Delta \eta) \tag{9.26d}$$

As it has been mentioned earlier $f''(0) = H(0) = \lambda$ is unknown. It must be determined such that the condition $f'(\infty) = G(\infty) = 1$ is satisfied. The condition at infinity is usually approximated at a finite value of η (around $\eta = 10$). The process of obtaining λ accurately involves iteration and may be calculated using the procedure described below.

For this purpose, consider Fig. 9.2(a) where the solutions of G versus η for two different values of $H(0)$ are plotted. The values of $G(\infty)$ are estimated from the G curves and are plotted in Fig. 9.2(b). The value of $H(0)$ now can be calculated by finding the value $\tilde{H}(0)$ at which the line 1–2 crosses the line $G(\infty) = 1$. By using similar traingles, it can be said that

$$\frac{\tilde{H}(0) - H(0)_1}{1 - G(\infty)_1} = \frac{H(0)_2 - H(0)_1}{G(\infty)_2 - G(\infty)_1}$$

By solving this, we get $\tilde{H}(0)$. Next we repeat the same calculation as above by using $\tilde{H}(0)$ and the better of the two initial values of $H(0)$. Thus we get another improvedvalue $\tilde{\tilde{H}}(0)$. This process may continue, that is, we use $\tilde{\tilde{H}}(0)$ and $\tilde{H}(0)$ as a pair of values to find more improved values for $H(0)$, and soforth. It should be always kept in mind that for each value of $H(0)$, the curve $G(\eta)$ versus η is to be examined to get the proper value of $G(\infty)$.

The functions $f(\eta)$, $f'(\eta) = G$ and $f''(\eta) = H$ are plotted in Fig. 9.3. The velocity components, u and v inside the boundary layer can be computed from
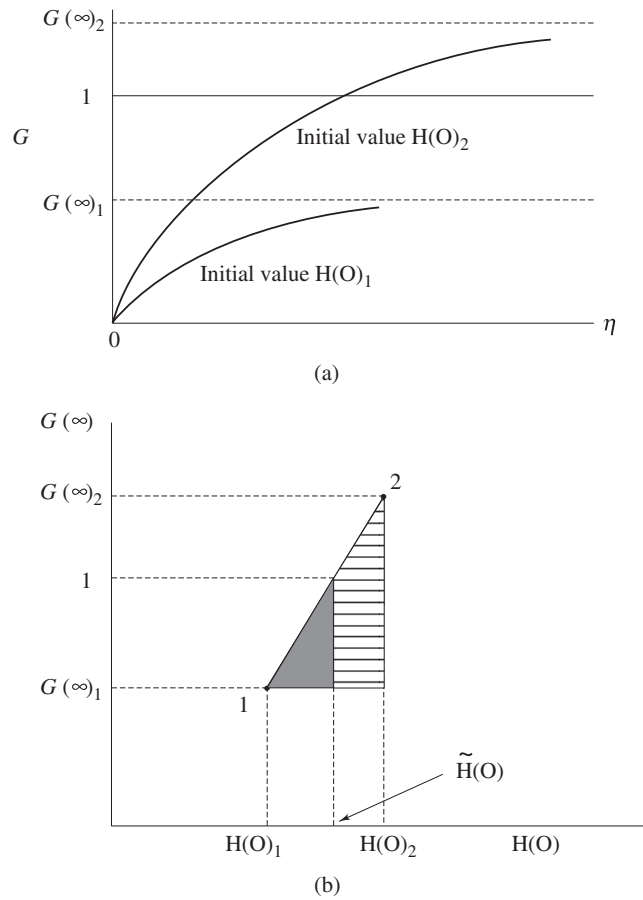
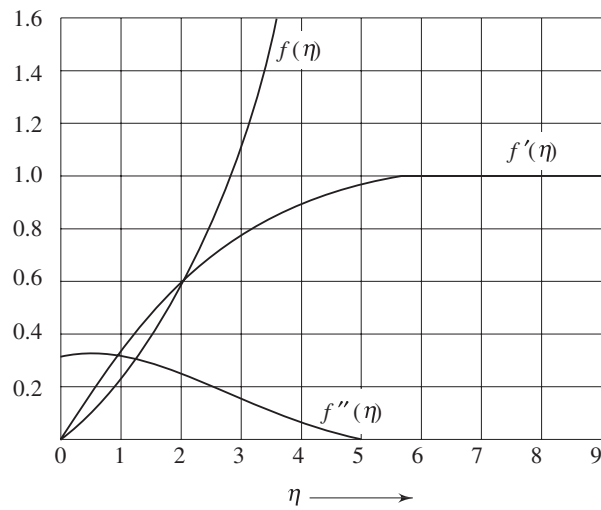(a)

(b)

Fig. 9.2   Correcting the initial guess for H(O)

Fig. 9.3   f, G, and H distribution in the boundary layer

Eqs (9.21a) and (9.21b) respectively. Measurements to test the accuracy of theoretical results were carried out by many scientists. In his experiments, J. Nikuradse, found excellent agreement with the theoretical results with respect to velocity distribution (u/U_∞) within the boundary layer of a stream of air on a flat plate. However, some values of the velocity profile shape $f'(\eta) = u/U_\infty = G$ and $f''(\eta) = H$ are given in Table 9.1.

Table 9.1 Blasius Velocity Profile $G = u/U_\infty$, f and H after Schlichting [2]

η	f	G	H
0	0	0	0.33206
0.2	0.00664	0.06641	0.33199
0.4	0.02656	0.13277	0.33147
0.8	0.10611	0.26471	0.32739
1.2	0.23795	0.39378	0.31659
1.6	0.42032	0.51676	0.29667
2.0	0.65003	0.62977	0.26675
2.4	0.92230	0.72899	0.22809
2.8	1.23099	0.81152	0.18401
3.2	1.56911	0.87609	0.13913
3.6	1.92954	0.92333	0.09809
4.0	2.30576	0.95552	0.06424
4.4	2.69238	0.97587	0.03897
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
8.8	7.07923	1.00000	0.00000

9.4 WALL SHEAR AND BOUNDARY LAYER THICKNESS

With the profile known, wall shear can be evaluated as

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0}$$

or

$$\tau_w = \mu\, U_\infty \left. \frac{\partial}{\partial \eta} f'(\eta) \cdot \frac{\partial \eta}{\partial y} \right|_{\eta=0}$$

or

$$\tau_w = \mu\, U_\infty \times 0.33206 \times \frac{1}{\sqrt{(\nu x)/U_\infty}}$$
$$[f''(0) = 0.33206 \text{ from Table 9.1}]$$

or

$$\tau_w = \frac{0.332\, \rho\, U_\infty^2}{\sqrt{\mathrm{Re}_x}} \tag{9.27a}$$

and the local skin friction coefficient is

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$$

Substituting from (9.27a) we get

$$C_{fx} = \frac{0.664}{\sqrt{Re_x}} \tag{9.27b}$$

In 1951, Liepmann and Dhawan [3], measured the shearing stress on a flat plate directly. Their results showed a striking confirmation of Eq. (9.27). Total frictional force per unit width for the plate of length L is

$$F = \int_0^L \tau_w \, dx$$

or

$$F = \int_0^L \frac{0.332\,\rho U_\infty^2}{\sqrt{\dfrac{U_\infty}{\nu}}} \frac{dx}{\sqrt{x}}$$

or

$$F = \left[\frac{0.332\,\rho U_\infty^2}{\sqrt{U_\infty/\nu}} \times \frac{x^{1/2}}{\left(\dfrac{1}{2}\right)} \right]_0^L$$

or

$$F = 0.664 \times \rho U_\infty^2 \sqrt{\frac{\nu L}{U_\infty}} \tag{9.28}$$

and the average skin friction coefficient is

$$\overline{C}_f = \frac{F}{\frac{1}{2}\left(\rho U_\infty^2 L\right)} = \frac{1.328}{\sqrt{Re_L}} \tag{9.29}$$

where, $Re_L = U_\infty L / \nu$.

Since u/U_∞ approaches 1.0 as $y \to \infty$, it is customary to select the boundary layer thickness δ as that point where u/U_∞ approaches 0.99. From Table 9.1, u/U_∞ reaches 0.99 at $\eta = 5.0$ and we can write

$$\delta \Big/ \sqrt{\left(\frac{\nu x}{U_\infty}\right)} \approx 5.0$$

or

$$\delta \approx 5.0 \sqrt{\left(\frac{\nu x}{U_\infty}\right)} = \frac{5.0\,x}{\sqrt{Re_x}} \tag{9.30}$$

However, the aforesaid definition of boundary layer thickness is somewhat arbitrary, a physically more meaningful measure of boundary layer estimation is expressed through displacement thickness.
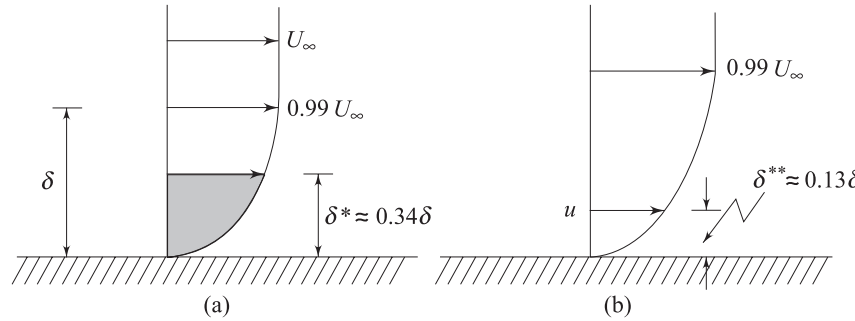
Fig. 9.4 (a) Displacement thickness (b) Momentum thickness

Displacement thickness (δ^*) is defined as the distance by which the external potential flow is displaced outwards due to the decrease in velocity in the boundary layer.

$$U_\infty \, \delta^* = \int_0^\infty (U_\infty - u) \, dy$$

Therefore,
$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy \tag{9.31}$$

Substituting the values of (u/U_∞) and η from Eqs (9.21a) and (9.19) into Eq. (9.31), we obtain

$$\delta^* = \sqrt{\frac{\nu x}{U_\infty}} \int_0^\infty (1 - f') \, d\eta = \sqrt{\frac{\nu x}{U_\infty}} \lim_{\eta \to \infty} [\eta - f(\eta)]$$

or
$$\delta^* = 1.7208 \sqrt{\frac{\nu x}{U_\infty}} = \frac{1.7208 \, x}{\sqrt{\mathrm{Re}_x}} \tag{9.32}$$

Following the analogy of the displacement thickness, a momentum thickness may be defined. *Momentum thickness* (δ^{**}) is defined as the loss of momentum in the boundary layer as compared with that of potential flow. Thus

$$\rho \, U_\infty^2 \, \delta^{**} = \int_0^\infty \rho u \, (U_\infty - u) \, dy$$

or
$$\delta^{**} = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy \tag{9.33}$$

With the substitution of (u/U_∞) and η from Eq. (9.21a) and (9.19), we can evaluate numerically the value of δ^{**} for a flat plate as

$$\delta^{**} = \sqrt{\frac{\nu x}{U_\infty}} \int_0^\infty f'(1 - f') \, d\eta$$

or
$$\delta^{**} = 0.664 \sqrt{\frac{\nu x}{U_\infty}} = \frac{0.664 \, x}{\sqrt{\mathrm{Re}_x}} \tag{9.34}$$

The relationships between δ, δ^* and δ^{**} have been shown in Fig. 9.4.

9.5 MOMENTUM-INTEGRAL EQUATIONS FOR BOUNDARY LAYER

If we are to employ boundary layer concepts in real engineering designs, we need to devise approximate methods that would quickly lead to an answer even if the accuracy is somewhat less. Karman and Pohlhausen devised a simplified method by satisfying only the boundary conditions of the boundary layer flow rather than satisfying Prandtl's differential equations for each and every particle within the boundary layer. We shall discuss this method herein.

Consider the case of steady, two-dimensional and incompressible flow, i.e. we shall refer to Eqs (9.10) to (9.14). Upon integrating the dimensional form of Eq. (9.10) with respect to $y = 0$ (wall) to $y = \delta$ (where δ signifies the interface of the free stream and the boundary layer), we obtain

$$\int_0^\delta \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) dy = \int_0^\delta \left(-\frac{1}{\rho} \frac{dp}{dx} + v \frac{\partial^2 u}{\partial y^2} \right) dy$$

or
$$\int_0^\delta u \frac{\partial u}{\partial x} dy + \int_0^\delta v \frac{\partial u}{\partial y} dy = \int_0^\delta -\frac{1}{\rho} \frac{dp}{dx} dy + \int_0^\delta v \frac{\partial^2 u}{\partial y^2} dy \qquad (9.35)$$

The second term of the left hand side can be expanded as

$$\int_0^\delta v \frac{\partial u}{\partial y} dy = [vu]_0^\delta - \int_0^\delta u \frac{\partial v}{\partial y} dy$$

or
$$\int_0^\delta v \frac{\partial u}{\partial y} dy = U_\infty v_\delta + \int_0^\delta u \frac{\partial u}{\partial x} dy \quad \left(\text{since } \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \right)$$

or
$$\int_0^\delta v \frac{\partial u}{\partial y} dy = -U_\infty \int_0^\delta \frac{\partial u}{\partial x} dy + \int_0^\delta u \frac{\partial u}{\partial x} dy \qquad (9.36)$$

Substituting Eq. (9.36) in Eq. (9.35) we obtain

$$\int_0^\delta 2u \frac{\partial u}{\partial x} dy - U_\infty \int_0^\delta \frac{\partial u}{\partial x} dy = -\int_0^\delta \frac{1}{\rho} \frac{dp}{dx} dy - v \frac{\partial u}{\partial y}\bigg|_{y=0} \qquad (9.37)$$

Substituting the relation between $\dfrac{dp}{dx}$ and the free stream velocity U_∞ for the inviscid zone in Eq. (9.37) we get

$$\int_0^\delta 2u \frac{\partial u}{\partial x} dy - U_\infty \int_0^\delta \frac{\partial u}{\partial x} dy - \int_0^\delta U_\infty \frac{dU_\infty}{dx} dy = -\left(\frac{\mu \frac{\partial u}{\partial y}\big|_{y=0}}{\rho} \right)$$

or
$$\int_0^\delta \left(2u \frac{\partial u}{\partial x} - U_\infty \frac{\partial u}{\partial x} - U_\infty \frac{dU_\infty}{dx} \right) dy = -\frac{\tau_w}{\rho}$$

which is reduced to

$$\int_0^\delta \frac{\partial}{\partial x} \{[u\,(U_\infty - u)]\,\mathrm{d}y\} + \frac{\mathrm{d}U_\infty}{\mathrm{d}x} \int_0^\delta (U_\infty - u)\,\mathrm{d}y = \frac{\tau_w}{\rho}$$

Since the integrals vanish outside the boundary layer, we are allowed to put $\delta = \infty$.

$$\int_0^\infty \frac{\partial}{\partial x} [u\,(U_\infty - u)]\mathrm{d}y + \frac{\mathrm{d}U_\infty}{\mathrm{d}x} \int_0^\infty (U_\infty - u)\,\mathrm{d}y = \frac{\tau_w}{\rho}$$

or $$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^\infty [u\,(U_\infty - u)]\mathrm{d}y + \frac{\mathrm{d}U_\infty}{\mathrm{d}x} \int_0^\infty (U_\infty - u)\,\mathrm{d}y = \frac{\tau_w}{\rho} \qquad (9.38)$$

Substituting Eq. (9.31) and (9.33) in Eq. (9.38) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[U_\infty^2\,\delta^{**}\right] + \delta^*\,U_\infty\,\frac{\mathrm{d}U_\infty}{\mathrm{d}x} = \frac{\tau_w}{\rho} \qquad (9.39)$$

Equation (9.39) is known as momentum integral equation for two dimensional incompressible laminar boundary layer. The same remains valid for turbulent boundary layers as well. Needless to say, the wall shear stress (τ_w) will be different for laminar and turbulent flows. The term $\left(U_\infty\,\dfrac{\mathrm{d}U_\infty}{\mathrm{d}x}\right)$ signifies

spacewise acceleration of the free stream. Existence of this term means the

presence of free stream pressure gradient in the flow direction. For example, we

get finite value of $\left(U_\infty\,\dfrac{\mathrm{d}U_\infty}{\mathrm{d}x}\right)$ outside the boundary layer in the entrance region

of a pipe or a channel. For external flows, the existence of $\left(U_\infty\,\dfrac{\mathrm{d}U_\infty}{\mathrm{d}x}\right)$ depends

on the shape of the body. During the flow over a flat plate, $\left(U_\infty\,\dfrac{\mathrm{d}U_\infty}{\mathrm{d}x}\right) = 0$ and

the momentum integral equation is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[U_\infty^2\,\delta^{**}\right] = \frac{\tau_w}{\rho} \qquad (9.40)$$

9.6 SEPARATION OF BOUNDARY LAYER

It has been observed that the flow is reversed at the vicinity of the wall under certain conditions. The phenomenon is termed as separation of boundary layer. Separation takes place due to excessive momentum loss near the wall in a boundary layer trying to move downstream against increasing pressure, i.e., $\mathrm{d}p/\mathrm{d}x > 0$, which is called *adverse pressure gradient*. Figure 9.5 shows the flow past a circular cylinder, in an infinite medium. Up to $\theta = 90°$, the flow area is like a constricted passage and the flow behaviour is like that of a nozzle. Beyond $\theta = 90°$ the flow area is diverged, therefore, the flow behaviour is much similar to

a diffuser. This dictates the inviscid pressure distribution on the cylinder which is shown by a firm line in Fig. 9.5. Here p_∞ and U_∞ are the pressure and velocity in the free stream and p is the local pressure on the cylinder.

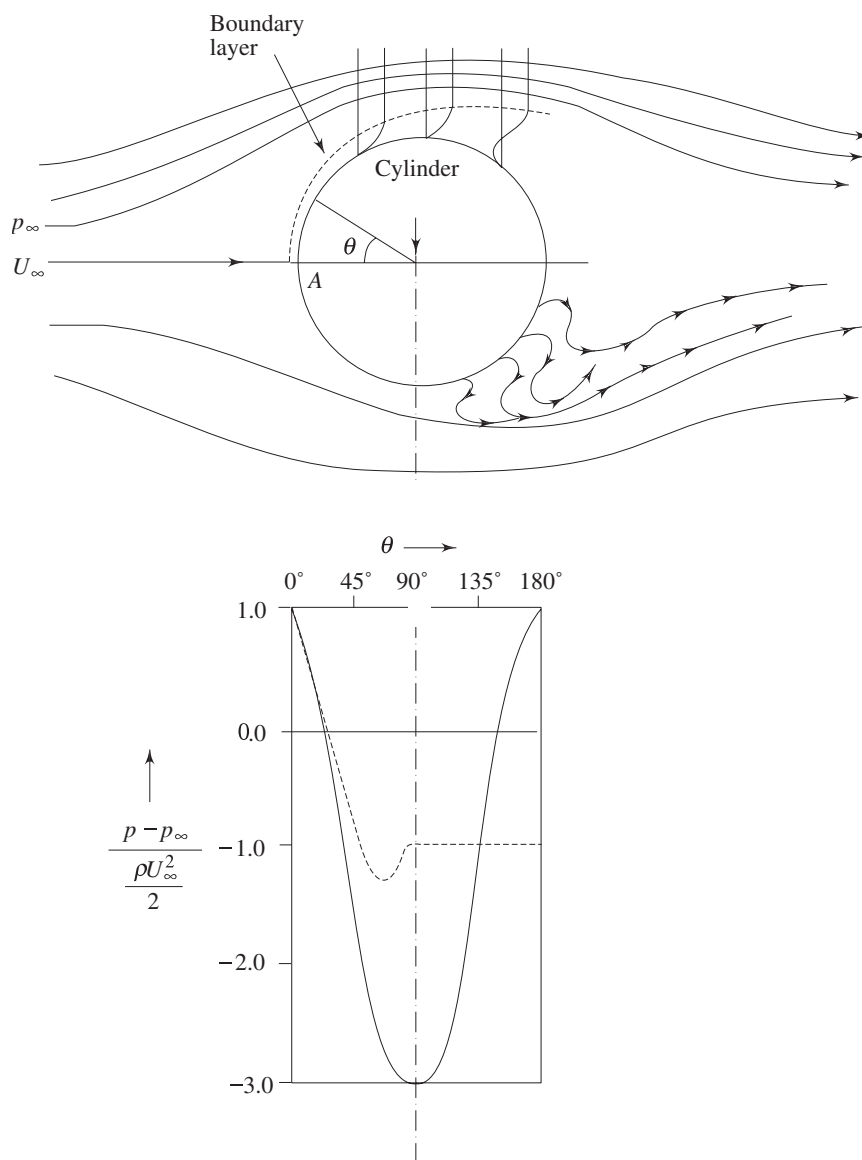**Fig. 9.5** Flow separation and formation of wake behind a circular cylinder

Consider the forces in the flow field. It is evident that in the inviscid region, the pressure force and the force due to streamwise acceleration are acting in the same direction (pressure gradient being negative/favourable) until $\theta = 90°$. Beyond $\theta = 90°$, the pressure gradient is positive or adverse. Due to the adverse pressure

gradient the pressure force and the force due to acceleration will be opposing each other in the inviscid zone of this part. So long as no viscous effect is considered, the situation does not cause any sensation. However, in the viscid region (near the solid boundary), up to $\theta = 90°$, the viscous force opposes the combined pressure force and the force due to acceleration. Fluid particles overcome this viscous resistance. Beyond $\theta = 90°$, within the viscous zone, the flow structure becomes different. It is seen that the force due to acceleration is opposed by both the viscous force and pressure force. Depending upon the magnitude of adverse pressure gradient, somewhere around $\theta = 90°$, the fluid particles, in the boundary layer are separated from the wall and driven in the upstream direction. However, the far field external stream pushes back these separated layers together with it and develops a broad pulsating wake behind the cylinder. Now let us look at the mathematical explanation of flow-separation. Following the foregoing observation, the point of separation may be defined as the limit between forward and reverse flow in the layer very close to the wall, i.e., at the point of separation

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0 \tag{9.41}$$

This means that the shear stress at the wall, $\tau_w = 0$. But at this point, the adverse pressure continues to exist and at the downstream of this point the flow acts in a reverse direction resulting in a back flow. We can also explain flow separation using the argument about the second derivative of velocity u at the wall. From the dimensional form of the momentum Eq. (9.10) at the wall, where $u = v = 0$, we can write

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = \frac{1}{\mu}\frac{dp}{dx} \tag{9.42}$$

Consider the situation due to a favourable pressure gradient were $\dfrac{dp}{dx} < 0$. From Eq. (9.42) we have, $(\partial^2 u/\partial y^2)_{\text{wall}} < 0$. As we proceed towards the free stream, the velocity u approaches U_∞ aysmptotically, so $\partial u/\partial y$ decreases at a continuously lesser rate in y direction. This means that $(\partial^2 u/\partial y^2)$ remains less than zero near the edge of the boundary layer. Finally it can be said that for a decreasing pressure gradient, the curvature of a velocity profile $(\partial^2 u/\partial y^2)$ is always negative as shown in (Fig. 9.6a). Next consider the case of adverse pressure gradient, $\partial p/\partial x > 0$. From Eq. (9.42), we observe that at the boundary, the curvature of the profile must be positive (since $\partial p/\partial x > 0$).

However, near the interface of boundary layer and free stream the previous argument regarding $\partial u/\partial y$ and $\partial^2 u/\partial y^2$ still holds good and the curvature is negative. Thus we observe that for an adverse pressure gradient, there must exist a point for which $\partial^2 u/\partial y^2 = 0$. This point is known as *point of inflection* of the velocity profile in the boundary layer as shown in Fig. 9.6b. However, point of separation means $\partial u/\partial y = 0$ at the wall. In addition, Eq. (9.42) depict $\partial^2 u/\partial y^2 > 0$ at the wall since separation can only occur due to adverse pressure gradient. But

we have already seen that at the edge of the boundary layer, $\partial^2 u/\partial y^2 < 0$. It is therefore, clear that if there is a point of separation, there must exist a point of inflection in the velocity profile.
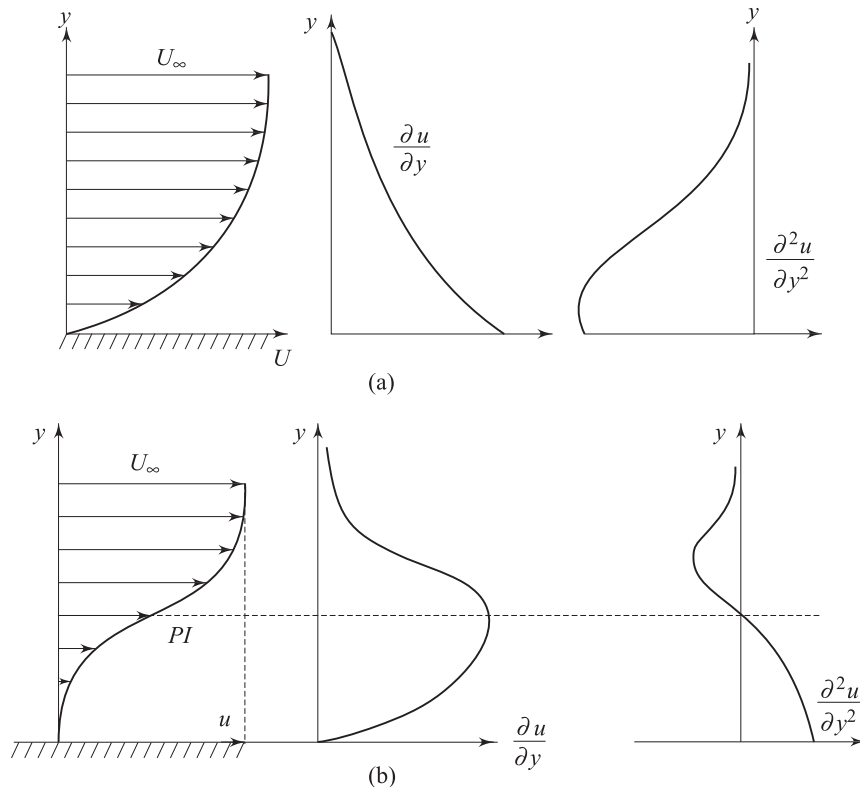

Fig. **9.6** Velocity distribution within a boundary layer

(a) Favourable pressure gradient, $\dfrac{dp}{dx} < 0$

(b) adverse pressure gradient, $\dfrac{dp}{dx} > 0$

Let us reconsider the flow past a circular cylinder and continue our discussion on the wake behind a cylinder. The pressure distribution which was shown by the firm line in Fig. 9.5 is obtained from the potential flow theory. However, somewhere near $\theta = 90°$ (in experiments it has been observed to be at $\theta = 81°$), the boundary layer detaches itself from the wall. Meanwhile, pressure in the wake remains close to separation-point-pressure since the eddies (formed as a consequence of the retarded layers being carried together with the upper layer through the action of shear) cannot convert rotational kinetic energy into pressure head. The actual pressure distribution is shown by the dotted line in Fig. 9.5. Since the wake zone pressure is less than that of the forward stagnation point (pressure at point A in Fig. 9.5), the cylinder experiences a drag force which is

basically attributed to the pressure difference. The drag force, brought about by the pressure difference is known as *form drag* whereas the shear stress at the wall gives rise to *skin friction drag*. Generally, these two drag forces together are responsible for resultant drag on a body.

9.7 KARMAN-POHLHAUSEN APPROXIMATE METHOD FOR SOLUTION OF MOMENTUM INTEGRAL EQUATION OVER A FLAT PLATE

The basic equation for this method is obtained by integrating the x direction momentum equation (boundary layer momentum equation) with respect to y from the wall (at $y = 0$) to a distance $\delta(x)$ which is assumed to be outside the boundary layer. With this notation, we can rewrite the Karman momentum integral equation (9.39) as

$$U_\infty^2 \frac{d\delta^{**}}{dx} + \left(2\delta^{**} + \delta^*\right) U_\infty \frac{dU_\infty}{dx} = \frac{\tau_w}{\rho} \tag{9.43}$$

The effect of pressure gradient is described by the second term on the left hand side. For pressure gradient surfaces in external flow or for the developing sections in internal flow, this term will be retained and will contribute to the pressure gradient. However, we assume a velocity profile which is a polynomial of $\eta = y/\delta$. As it has been seen earlier, η is a form of similarity variable. This implies that with the growth of boundary layer as distance x varies from the leading edge, the velocity profile (u/U_∞) remains geometrically similar. We choose a velocity profile in the form

$$\frac{u}{U_\infty} = a_0 + a_1 \, \eta + a_2 \, \eta^2 + a_3 \, \eta^3 \tag{9.44}$$

In order to determine the constants a_0, a_1, a_2 and a_3 we shall prescribe the following boundary conditions

at $\qquad y = 0, \quad u = 0 \qquad$ or $\qquad$ at $\eta = 0, \dfrac{u}{U_\infty} = 0$ $\qquad$ (9.45a)

at $\qquad y = 0, \dfrac{\partial^2 u}{\partial y^2} = 0 \qquad$ or $\qquad$ at $\eta = 0, \dfrac{\partial^2}{\partial \eta^2}(u/U_\infty) = 0$ $\qquad$ (9.45b)

at $\qquad y = \delta, u = U_\infty \qquad$ or $\qquad$ at $\eta = 1, \dfrac{u}{U_\infty} = 1$ $\qquad$ (9.45c)

at $\qquad y = \delta, \dfrac{\partial u}{\partial y} = 0 \qquad$ or $\qquad$ at $\eta = 1, \dfrac{\partial(u/U_\infty)}{\partial \eta} = 0$ $\qquad$ (9.45d)

These requirements will yield

$$a_0 = 0, \quad a_2 = 0, \quad a_1 + 3a_3 = 0 \quad \text{and} \quad a_1 + a_3 = 1$$

Finally, we obtain the following values for the coefficients in Eq. (9.44),

$$a_0 = 0, \quad a_1 = \frac{3}{2}, \quad a_2 = 0 \quad \text{and} \quad a_3 = -\frac{1}{2}$$

and the velocity profile becomes

$$\frac{u}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \qquad (9.46)$$

For flow over a flat plate, $\dfrac{dp}{dx} = 0$, hence $U_\infty \dfrac{dU_\infty}{dx} = 0$ and the governing Eq. (9.43) reduces to

$$\frac{d\delta^{**}}{dx} = \frac{\tau_w}{\rho U_\infty^2} \qquad (9.47)$$

Again from Eq. (9.33), the momentum thickness is

$$\delta^{**} = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

or

$$\delta^{**} = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

or

$$\delta^{**} = \delta \int_0^1 \left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right)\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right) d\eta$$

or

$$\delta^{**} = \frac{39}{280}\delta$$

The wall shear stress is given by

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

or

$$\tau_w = \mu \left[\frac{\partial}{\delta \partial \eta}\left\{U_\infty\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\right\}\right]_{\eta=0}$$

or

$$\tau_w = \frac{3\mu U_\infty}{2\delta}$$

Substituting the values of δ^{**} and τ_w in Eq. (9.47) we get,

$$\frac{39}{280}\frac{d\delta}{dx} = \frac{3\mu U_\infty}{2\delta \rho U_\infty^2}$$

or

$$\int \delta\, d\delta = \int \frac{140}{13}\frac{\mu}{\rho U_\infty}\, dx + C_1$$

or

$$\frac{\delta^2}{2} = \frac{140}{13}\frac{vx}{U_\infty} + C_1 \qquad (9.48)$$

where C_1 is any arbitrary unknown constant.

The condition at the leading edge (at $x = 0$, $\delta = 0$) yields

$$C_1 = 0$$

Finally we obtain,

$$\delta^2 = \frac{280}{13}\frac{vx}{U_\infty} \tag{9.49}$$

or

$$\delta = 4.64\sqrt{\frac{vx}{U_\infty}}$$

or

$$\delta = \frac{4.64x}{\sqrt{\mathrm{Re}_x}} \tag{9.50}$$

This is the value of boundary layer thickness on a flat plate. Although, the method is an approximate one, the result is found to be reasonably accurate. The value is slightly lower than the exact solution of laminar flow over a flat plate given by Eq. (9.30). As such, the accuracy depends on the order of the velocity profile. It may be mentioned that instead of Eq. (9.44), we can as well use a fourth order polynomial as

$$\frac{u}{U_\infty} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 \tag{9.51}$$

In addition to the boundary conditions in Eq. (9.45), we shall require another boundary condition

at $\qquad y = \delta, \dfrac{\partial^2 u}{\partial y^2} = 0 \quad$ or $\quad$ at $\eta = 1, \dfrac{\partial^2\,(u/U_\infty)}{\partial\eta^2} = 0$

To determine the constants as $a_0 = 0$, $a_1 = 2$, $a_2 = 0$, $a_3 = -2$ and $a_4 = 1$. Finally the velocity profile will be

$$\frac{u}{U_\infty} = 2\eta - 2\eta^3 + \eta^4$$

Subsequently, for a fourth order profile the growth of boundary layer is given by

$$\delta = \frac{5.83\,x}{\sqrt{\mathrm{Re}_x}} \tag{9.52}$$

This is also very close to the value of the exact solution.

9.8 INTEGRAL METHOD FOR NON-ZERO PRESSURE GRADIENT FLOWS

A wide variety of "integral methods" in this category have been discussed by Rosenhead [4]. The Thwaites method [5] is found to be a very elegant method, which is an extension of the method due to Holstein and Bohlen [6]. We shall discuss the Holstein-Bohlen method in this section.

This is an approximate method for solving boundary layer equations for two-dimensional generalized flow. The integrated Eq. (9.39) for laminar flow with pressure gradient can be written as

$$\frac{d}{dx}[U^2\delta^{**}]+\delta^{**}U\frac{dU}{dx}=\frac{\tau_w}{\rho}$$

or

$$U^2\frac{d\delta^{**}}{dx}+(2\delta^{**}+\delta^*)U\frac{dU}{dx}=\frac{\tau_w}{\rho} \tag{9.53}$$

The velocity profile the boundary layer is considered to be a fourth-order polynomial in terms of the dimensionless distance $\eta=y/\delta$, and is expressed as

$$u/U=a\eta+b\eta^2+c\eta^3+d\eta^4$$

The boundary conditions are

$$\eta=0:u=0,\ v=0 \qquad\qquad \frac{v}{\delta^2}\frac{\partial^2 u}{\partial\eta^2}=\frac{1}{\rho}\frac{dp}{dx}=-U\frac{dU}{dx}$$

$$\eta=1:u=U, \qquad\qquad \frac{\partial u}{\partial\eta}=0,\ \frac{\partial^2 u}{\partial\eta^2}=0$$

A dimensionless quantity, known as shape factor is introduced as

$$\lambda=\frac{\delta^2}{v}\frac{dU}{dx} \tag{9.54}$$

The following relations are obtained

$$a=2+\frac{\lambda}{6},\quad b=-\frac{\lambda}{2},\quad c=-2+\frac{\lambda}{2},\quad d=1-\frac{\lambda}{6}$$

Now, the velocity profile can be expressed as

$$u/U=F(\eta)+\lambda G(\eta), \tag{9.55}$$

where

$$F(\eta)=2\eta-2\eta^3+\eta^4,\quad G(\eta)=\frac{1}{6}\eta(1-\eta)^3$$

The shear stress $\tau_w=\mu(\partial u/\partial y)_{y=0}$ is given by

$$\frac{\tau_w\delta}{\mu U}=2+\frac{\lambda}{6} \tag{9.56}$$

We use the following dimensionless parameters,

$$L=\frac{\tau_w\delta^{**}}{\mu U} \tag{9.57}$$

$$K=\frac{(\delta^{**})^2}{v}\frac{dU}{dx} \tag{9.58}$$

$$H=\delta^*/\delta^{**} \tag{9.59}$$

The integrated momentum Eq. (9.53) reduces to

$$U\frac{d\delta^{**}}{dx}+\delta^{**}(2+H)\frac{dU}{dx}=\frac{vL}{\delta^{**}}$$

or

$$U \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{(\delta^{**})^2}{\nu} \right] = 2[L - K\,(H+2)] \tag{9.60}$$

The parameter L is related to the skin friction and K is linked to the pressure gradient. If we take K as the independent variable, L and H can be shown to be the functions of K since

$$\frac{\delta^*}{\delta} = \int_0^1 [1 - F(\eta) - \lambda G(\eta)]\mathrm{d}\eta = \frac{3}{10} - \frac{\lambda}{120} \tag{9.61}$$

$$\frac{\delta^{**}}{\delta} = \int_0^1 (F(\eta) + \lambda G(\eta))\,(1 - F(\eta) - \lambda G(\eta))\mathrm{d}\eta$$

$$= \frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072} \tag{9.62}$$

$$K = \frac{[\delta^{**}]^2}{\delta^2}\lambda = \lambda\left(\frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072} \right)^2 \tag{9.63}$$

Therefore,

$$L = \left(2 + \frac{\lambda}{6} \right)\frac{\delta^{**}}{\delta} = \left(2 + \frac{\lambda}{6} \right)\left(\frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072} \right) = f_1(K)$$

$$H = \frac{\delta^*}{\delta^{**}} = \frac{(3/10) - (\lambda/120)}{(37/315) - (\lambda/945) - (\lambda^2/9072)} = f_2(K)$$

The right-hand side of Eq. (9.60) is thus a function of K alone. Walz [7] pointed out that this function can be approximated with a good degree of accuracy by a linear function of K so that

$$2[L - K(H+2)] = a - bK$$

Equation (9.60) can now be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{U[\delta^{**}]^2}{\nu} \right) = a - (b-1)\,\frac{U[\delta^{**}]^2}{\nu}\,\frac{1}{U}\frac{\mathrm{d}U}{\mathrm{d}x}$$

Solution of this differential equation for the dependent variable $(U[\delta^{**}]^2/\nu)$ subject to the boundary condition $U = 0$ when $x = 0$, gives

$$\frac{U[\delta^{**}]^2}{\nu} = \frac{a}{U^{b-1}}\int_0^x U^{b-1}\,\mathrm{d}x$$

With $a = 0.47$ and $b = 6$, the approximation is particularly close between the stagnation point and the point of maximum velocity. Finally the value of the dependent variable is

$$[\delta^{**}]^2 = \frac{0.47\nu}{U^6} \int_0^x U^5 \, dx \tag{9.64}$$

By taking the limit of Eq. (9.64), according to L'Hospital's rule, it can be shown that

$$[\delta^{**}]^2|_{x=0} = 0.47\,\nu/6U'(0)$$

This corresponds to $K = 0.0783$. It may be mentioned that $[\delta^{**}]$ is not equal to zero at the stagnation point. If $([\delta^{**}]^2/\nu)$ is determined from Eq. (9.64), $K(x)$ can be obtained from Eq. (9.58). Table 9.2 gives the necessary parameters for obtaining results, such as velocity profile and shear stress τ_w. The approximate method can be applied successfully to a wide range of problems.

Table 9.2   Auxiliary functions after Holstein and Bohlen [6]

λ	K	$f_1(K)$	$f_2(K)$
12	0.0948	2.250	0.356
10	0.0919	2.260	0.351
8	0.0831	2.289	0.340
7.6	0.0807	2.297	0.337
7.2	0.0781	2.305	0.333
7.0	0.0767	2.309	0.331
6.6	0.0737	2.318	0.328
6.2	0.0706	2.328	0.324
5.0	0.0599	2.361	0.310
3.0	0.0385	2.427	0.283
1.0	0.0135	2.508	0.252
0	0	2.554	0.235
− 1	− 0.0140	2.604	0.217
− 3	− 0.0429	2.716	0.179
− 5	− 0.0720	2.847	0.140
− 7	− 0.0999	2.999	0.100
− 9	− 0.1254	3.176	0.059
− 11	− 0.1474	3.383	0.019
− 12	− 0.1567	3.500	0

As mentioned earlier, K and λ are related to the pressure gradient and the shape factor. Introduction of K and λ in the integral analysis enables extension of Karman-Pohlhausen method for solving flows over curved geometry. However, the analysis is not valid for the geometries, where $\lambda < -12$ and $\lambda > +12$.

9.9   ENTRY FLOW IN A DUCT

Growth of boundary layer has a remarkable influence on flow through a constant area duct or pipe. Let us consider a flow entering a pipe with uniform velocity.

The boundary layer starts growing on the wall at the entrance of the pipe. Gradually it becomes thicker in the downstream and the flow becomes fully developed when the boundary layers from the wall meet at the axis of the pipe (Fig. 9.7). The velocity profile is nearly rectangular at the entrance and it gradually changes to a parabolic profile at the fully developed region. Before the boundary layers from the periphery meet at the axis, there prevails a core region which is uninfluenced by viscosity. Since the volume-flow must be same for every section and the boundary-layer thickness increases in the flow direction, the inviscid core accelerates, and there is a corresponding fall in pressure. It can be shown that for laminar incompressible flows, the velocity profile approaches the parabolic profile through a distance *Le* from the entry of the pipe, which is given by

$$\frac{Le}{D} \approx 0.05 \text{ Re}, \qquad \text{where Re} = \frac{U_{av}D}{\nu}$$

For a Reynolds number of 2000, this distance, which is often referred to as the entrance length is about 100 pipe-diameters. For turbulent flows, the entrance region is shorter, since the turbulent boundary layer grows faster.

At the entrance region, the velocity gradient is steeper at the wall, causing a higher value of shear stress as compared to a developed flow. In addition, momentum flux across any section in the entrance region is higher than that typically at the inlet due to the change in shape of the velocity profile. Arising out of these, an additional pressure drop is brought about at the entrance region as compared to the pressure drop in the fully developed region.
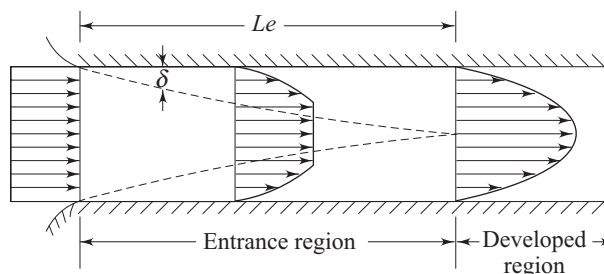

Fig. 9.7 Development of boundary layer in the entrance region of a duct

9.10 CONTROL OF BOUNDARY LAYER SEPARATION

It has already been seen that the total drag on a body is attributed to form drag and skin friction drag. In some flow configurations, the contribution of form drag becomes significant. In order to reduce the form drag, the boundary layer separation should be prevented or delayed so that somewhat better pressure recovery takes place and the form drag is reduced considerably. There are some popular methods for this purpose which are stated as follows.

(i) By giving the profile of the body a streamlined shape as shown in Fig. 9.8. This has an elongated shape in the rear part to reduce the magnitude of the pressure gradient. The optimum contour for a streamlined body is the one for which the wake zone is very narrow and the form drag is minimum.
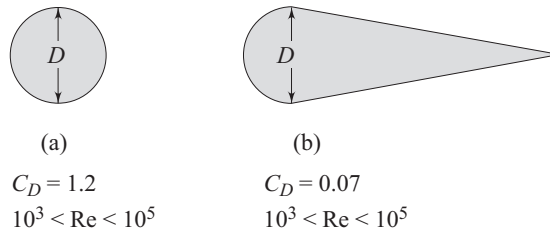

(a)

$C_D = 1.2$

$10^3 < \mathrm{Re} < 10^5$

(b)

$C_D = 0.07$

$10^3 < \mathrm{Re} < 10^5$

Fig. 9.8 Reduction of drag coefficient (C_D) by giving the profile a streamlined shape

(ii) The injection of fluid through porous wall can also control the boundary layer separation. This is generally accomplished by blowing high energy fluid particles tangentially from the location where separation would have taken place otherwise. This is shown in Fig. 9.9. The injection of fluid promotes turbulence and thereby increases skin friction. But the form drag is reduced considerably due to suppression of flow separation and this reduction can be of significant magnitude so as to ignore the enhanced skin friction drag.
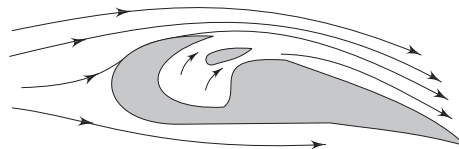

Fig. 9.9 Boundary layer control by blowing

9.11 MECHANICS OF BOUNDARY LAYER TRANSITION

One of the interesting problems in fluid mechanics is the physical mechanism of transition from laminar to turbulent flow. The problem evolves about the generation of both steady and unsteady vorticity near a body, its subsequent molecular diffusion, its kinematic and dynamic convection and redistribution downstream, and the resulting feedback on the velocity and pressure fields near the body. We can perhaps realise the complexity of the transition problem by examining the behaviour of a real flow past a cylinder.

Figure 9.10 (a) shows the flow past a cylinder for a very low Reynolds number (~1). The flow smoothly divides and reunites around the cylinder. At a Reynolds number of about 4, the flow separates in the downstream and the wake is formed

by two symmetric eddies. The eddies remain steady and symmetrical but grow in size up to a Reynolds number of about 40 as shown in Fig. 9.10(b).

When the Reynolds number crosses 40, oscillation in the wake induces asymmetry and finally the wake starts shedding vortices into the stream. This situation is termed as onset of periodicity as shown in Fig. 9.10(c) and the wake keeps on undulating up to a Reynolds number of 90. As the Reynolds number further increases, the eddies are shed alternately from a top and bottom of the cylinder and the regular pattern of alternately shed clockwise and counter-clockwise vortices form *Von Karman vortex street* as in Fig. 9.10(d). However, periodicity is eventually induced in the flow field with the vortex-shedding phenomenon. The periodicity is characterised by the frequency of vortex shedding f. In non-dimensional form, the vortex shedding frequency is expressed as fD/U_{ref}, known as the *Strouhal number* named after V. Strouhal, a German physicist who experimented with wires singing in the wind. The Strouhal number shows a slight but continuous variation with Reynolds number around a value of 0.21. At about Re = 500, multiple frequencies start showing up and the wake tends to become turbulent.
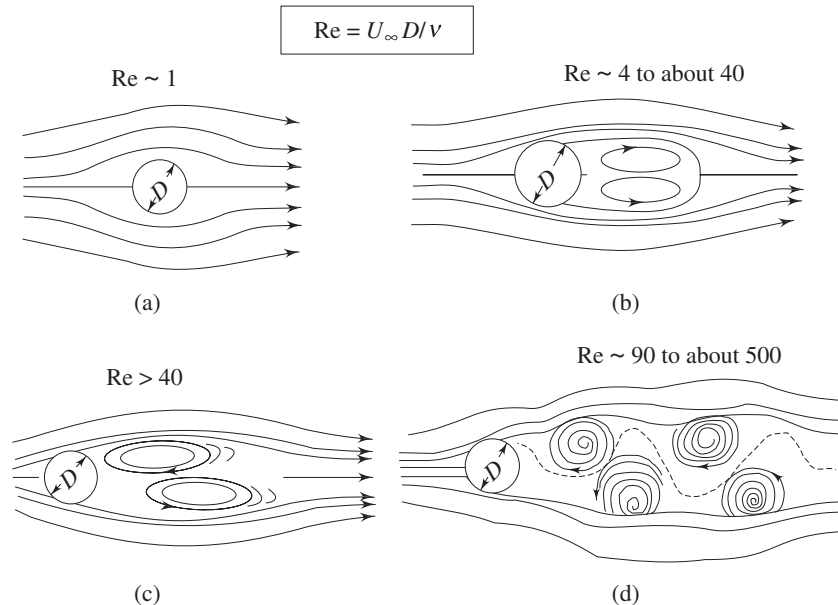
$$Re = U_\infty D/\nu$$

Re ~ 1

Re ~ 4 to about 40

(a)

(b)

Re > 40

Re ~ 90 to about 500

(c)

(d)

Fig. 9.10 Influence of Reynolds number on wake-zone aerodynamics

An understanding of the transitional flow processes will help in practical problems either by improving procedures for predicting positions or for determining methods of advancing or retarding the transition position.

The critical value at which the transition occurs in pipe flow is $Re_{cr} = 2300$. The actual value depends upon the disturbance in flow. Some experiments have shown the critical Reynolds number to reach as high as 40,000. The precise upper bound is not known, but the lower bound appears to be $Re_{cr} = 2300$. Below this value, the flow remains laminar even when subjected to strong disturbances.

For $2300 \leq \mathrm{Re}_{cr} \leq 2600$, the flow alternates randomly between laminar and partially turbulent. Near the centerline, the flow is more laminar than turbulent, whereas near the wall, the flow is more turbulent than laminar. For flow over a flat plate, turbulent regime is observed between Reynolds numbers ($U_\infty x/\nu$) of 3.5×10^5 and 10^6.

9.12 SEVERAL EVENTS OF TRANSITION

Transitional flow consists of several events as shown in Fig. 9.11. Let us consider the events one after another.

1. Region of instability of small wavy disturbances   Consider a laminar flow over a flat plate aligned with the flow direction (Fig. 9.11). It has been seen that in the presence of an adverse pressure gradient, at a high Reynolds number (water velocity approximately 9-cm/sec), two-dimensional waves appear. These waves can be made visible by a method known as tellurium method. In 1929, Tollmien and Schlichting predicted that the waves would form and grow in the boundary layer. These waves are called *Tollmien-Schlichting* wave.

2. Three-dimensional waves and vortex formation   Disturbances in the free stream or oscillations in the upstream boundary layer can generate wave growth, which has a variation in the spanwise direction. This leads an initially two-dimensional wave to a three-dimensional form. In many such transitional flows, periodicity is observed in the spanwise direction. This is accompanied by the appearance of vortices whose axes lie in the direction of flow.

3. Peak-Valley development with streamwise vortices   As the three-dimensional wave propagates downstream, the boundary layer flow develops into a complex streamwise vortex system. However, within this vortex system, at some spanwise location, the velocities fluctuate violently. These locations are called peaks and the neighbouring locations of the peaks are valleys (Fig. 9.12).

4. Vorticity concentration and shear layer development   At the spanwise locations corresponding to the peak, the instantaneous streamwise velocity profiles demonstrate the following. Often, an inflexion is observed on the velocity profile. The inflectional profile appears and disappears once after each cycle of the basic wave.

5. Breakdown   The instantaneous velocity profiles produce high shear in the outer region of the boundary layer. The velocity fluctuations develop from the shear layer at a higher frequency than that of the basic wave. However, these velocity fluctuations have a strong ability to amplify any slight three-dimensionality, which is already present in the flow field. As a result, a staggered vortex pattern evolves with the streamwise wavelength twice the wavelength of *Tollmien-Schlichting wavelength*. The spanwise wavelength of these structures is about one-half of the streamwise value. This is known as breakdown. Klebanoff et al. [8] refer to the high frequency fluctuations as hairpin eddies.

6. *Turbulent-spot development*   The hairpin-eddies travel at a speed grater than that of the basic (primary) waves. As they travel downstream, eddies spread in the spanwise direction and towards the wall. The vortices begin a cascading breakdown into smaller vortices. In such a fluctuating state, intense local changes occur at random locations in the shear layer near the wall in the form of turbulent spots. Each spot grows almost linearly with the downstream distance. The creation of spots is considered as the main event of transition.
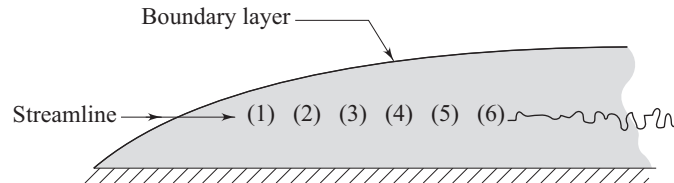

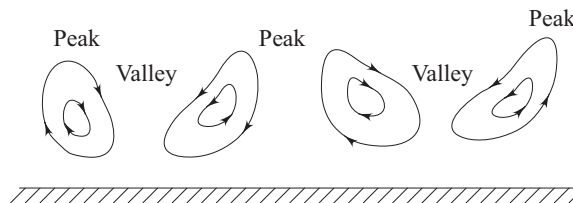**Fig. 9.11   Sequence of event involved in transition**

Fig. 9.12   Cross-stream view of the streamwise vortex system

Summary

- The boundary layer is the thin layer of fluid adjacent to the solid surface. Phenomenologically, the effect of viscosity is very prominent within this layer.
- The main-stream velocity undergoes a change from zero at the solid-surface to the full magnitude through the boundary layer.
- Effectively, the boundary layer theory is a complement to the inviscid flow theory.
- The governing equation for the boundary layer can be obtained through correct reduction of the *Navier-Stokes equations* within the thin layer referred above. There is no variation in pressure in *y* direction within the boundary layer.
- The pressure is impressed on the boundary layer by the outer inviscid flow which can be calculated using *Bernoulli's equation*.
- The boundary layer equation is a second order non-linear partial differential equation. The exact solution of this equation is known as *similarity solution*. For the flow over a flat plate, the similarity solution is often referred to as *Blasius solution*. Complete analytical treatment of this solution is beyond the scope of this text. However, the momentum integral equation can be derived from the boundary layer equation which is amenable to analytical treatment.

- The solutions of the momentum integral equation are called approximate solutions of the boundary layer equation.
- The boundary layer equations are valid up to the point of separation. At the point of separation, the flow gets detached from the solid surface due to excessive adverse pressure gradient.
- Beyond the point of separation, the flow reversal produces eddies. During flow past bluff-bodies, the desired pressure recovery does not take place in a separated flow and the situation gives rise to *pressure drag* or *form drag*.

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Solved Examples

Example 9.1   Water flows over a flat plate at a free stream velocity of 0.15 m/s. There is no pressure gradient and laminar boundary layer is 6 mm thick. Assume a sinusoidal velocity profile

$$\frac{u}{U_\infty} = \sin \frac{\pi}{2}\left(\frac{y}{\delta}\right)$$

For the flow conditions stated above, calculate the local wall shear stress and skin friction coefficient.

$$[\mu = 1.02 \times 10^{-3} \text{ kg/ms}, \rho = 1000 \text{ kg/m}^3]$$

Solution
$$\tau = \mu \frac{\partial u}{\partial y} = \frac{\mu U_\infty}{\delta} \cdot \frac{\partial (u/U_\infty)}{\partial (y/\delta)}$$

$$= \frac{\mu U_\infty}{\delta} \frac{\pi}{2} \cos\left(\frac{\pi}{2} \cdot \frac{y}{\delta}\right) = \frac{1.57 \, \mu U_\infty}{\delta} \cos\left(\frac{\pi}{2} \cdot \frac{y}{\delta}\right)$$

$$\tau_w = \tau|_{y=0} = \frac{1.57 \, \mu U_\infty}{\delta}$$

or
$$\tau_w = \frac{1.57 \times 1.02 \times 10^{-3} \times 0.15}{6 \times 10^{-3}} = 0.04 \text{ N/m}^2$$

$$C_f = \frac{\tau_w}{\frac{1}{2} \rho U_\infty^2} = \frac{2 \times 0.04}{1000 \times (0.15)^2} = 3.5 \times 10^{-3}$$

Example 9.2 Air at standard conditions flows over a flat plate. The freestream speed is 3 m/sec. Find δ and τ_w at $x = 1$ m from the leading edge (assume a cubic velocity profile). For air, $\nu = 1.5 \times 10^{-5}$ m²/sec and $\rho = 1.23$ kg/m³.

Solution Applying the results developed in section 9.7 for cubic velocity profile and the growth of boundary layer, we can write

$$\frac{U}{U_\infty} = \frac{3}{2} \eta - \frac{1}{2} \eta^3, \text{ where } \eta = \frac{y}{\delta} \text{ at any } x$$

and
$$\frac{\delta}{x} = \frac{4.64}{\sqrt{\text{Re}_x}}$$

For air with $\nu = 1.5 \times 10^{-5}$ m²/s, the local Reynolds number at x is

$$\text{Re}_x = \frac{U_\infty x}{\nu} = \frac{3 \times 1}{1.5 \times 10^{-5}} = 2 \times 10^5$$

$$\delta = \frac{4.64 \, x}{\sqrt{\text{Re}_x}} = \frac{4.64 \times 1}{\sqrt{2 \times 10^5}} \text{ m} = 0.0103 \text{ m} = 10.3 \text{ mm}$$

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\mu U_\infty}{\delta} \cdot \frac{d}{d\eta}\left[\frac{3}{2}\eta - \frac{1}{2}\eta^3\right]_{\eta=0}$$

$$\tau_w = \frac{3 \mu U_\infty}{2 \delta} = \frac{3 \rho \nu U_\infty}{2 \delta}$$

or
$$\tau_w = \frac{3 \times 1.23 \times 1.5 \times 10^{-5} \times 3}{2 \times 0.0103}$$

$$\tau_w = 8.06 \times 10^{-3} \text{ N/m}^2$$

Example 9.3 Air moves over a flat plate with a uniform free stream velocity of 10 m/s. At a position 15 cm away from the front edge of the plate, what is the boundary layer thickness? Use a parabolic profile in the boundary layer. For air, $\nu = 1.5 \times 10^{-5}$ m²/s and $\rho = 1.23$ kg/m³.

Solution For a parabolic profile let us take

$$\frac{u}{U_\infty} = a + by + cy^2$$

The boundary conditions are

at $\qquad\qquad y = 0, \qquad u = 0$

at $\qquad\qquad y = \delta, \qquad u = U_\infty$

at $\qquad\qquad y = \delta, \qquad \dfrac{\partial u}{\partial y} = 0$

Evaluating the constants we get

$$\frac{u}{U_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 = 2\eta - \eta^2$$

Now $\qquad\qquad \tau_w = \mu \left.\dfrac{\partial u}{\partial y}\right|_{y=0} = \dfrac{\mu U_\infty}{\delta} \cdot \left.\dfrac{\partial(u/U_\infty)}{\partial(y/\delta)}\right|_{\eta=0}$

$$= \frac{\mu U_\infty}{\delta} \cdot \left.\frac{\mathrm{d}(2\eta - \eta^2)}{\mathrm{d}\eta}\right|_{\eta=0} = \frac{2\mu U_\infty}{\delta}$$

Applying momentum integral equation

$$\tau_w = \rho U_\infty^2 \frac{\mathrm{d}\delta}{\mathrm{d}x} \int_0^1 \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)\mathrm{d}\eta$$

$$\frac{2\mu U_\infty}{\delta} = \rho U_\infty^2 \frac{\mathrm{d}\delta}{\mathrm{d}x} \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\,\mathrm{d}\eta$$

$$\frac{2\mu U_\infty}{\delta \rho U_\infty^2} = \frac{\mathrm{d}\delta}{\mathrm{d}x} \int_0^1 (2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\,\mathrm{d}\eta$$

or $\qquad\qquad \dfrac{2\mu}{\delta \rho U_\infty} = \dfrac{2}{15}\dfrac{\mathrm{d}\delta}{\mathrm{d}x}$

or $\qquad\qquad \delta\,\mathrm{d}\,\delta = \dfrac{15\mu}{\rho U_\infty}\,\mathrm{d}x$

$$\frac{\delta^2}{2} = \frac{15\mu}{\rho U_\infty}x + C$$

It is assumed that at $x = 0$, $\delta = 0$ which yields $C = 0$. Thus

$$\delta = \sqrt{\frac{30\mu}{\rho U_\infty}x}$$

or $\qquad\qquad \dfrac{\delta}{x} = \sqrt{\dfrac{30\mu}{\rho U_\infty x}} = \dfrac{5.48}{\sqrt{\mathrm{Re}_x}}$

In this problem, $\qquad \mathrm{Re}_x = \dfrac{10 \times 15 \times 10^{-2}}{1.5 \times 10^{-5}} = 1 \times 10^5$

$$\delta = \frac{5.48}{\sqrt{Re_x}} \times 15 \text{ cm} = 0.259 \text{ cm}$$

or $$\delta = 2.59 \text{ mm}$$

Example 9.4 Air moves over a 10 m long flat plate. The transition from laminar to turbulent flow takes place between Reynolds numbers of 2.5×10^6 and 3.6×10^6. What are the minimum and maximum distance from the front edge of the plate along which one expect laminar flow in the boundary layer? The free stream velocity is 30 m/s and $v = 1.5 \times 10^{-5}$ m²/s.

Solution We can see that the range of Reynolds numbers for laminar flow is 2.5×10^6 to 3.6×10^6

For the lower limit,

$$2.15 \times 10^6 = \frac{30 \, x}{1.5 \times 10^{-5}}$$

or $$x_{min} = 1.075 \text{ m}$$

for the upper limit,

$$3.6 \times 10^{-6} = \frac{30 \, x}{1.5 \times 10^{-5}}$$

or $$x_{max} = 1.8 \text{ m}$$

Example 9.5 Water at 15 °C flows over a flat plate at a speed of 1.2 m/s. The plate is 0.3 m long and 2 m wide. The boundary layer on each surface of the plane is laminar. Assume the velocity profile is approximated by a linear expression for which $\dfrac{\delta}{x} = \dfrac{3.46}{\sqrt{Re_x}}$.

Determine the drag force on the plate. For water $v = 1.1 \times 10^{-6}$ m²/s, $\rho = 1000$ kg/m³.

Solution On a flat plate, the drag is due to skin friction acting on each side of the plate

$$F_D = 2 \int_0^L \tau_{wx} \, b \, dx$$

For linear profile $\dfrac{u}{U_\infty} = \dfrac{y}{\delta}$ and $\tau_w = \mu \dfrac{\partial u}{\partial y}\Big|_{y=0}$

or $$\tau_w = \frac{\mu U_\infty}{\delta} \cdot \frac{\partial (U/U_\infty)}{\partial (y/\delta)}\Big|_{y=0} = \frac{\mu U_\infty}{\delta}$$

$$F_D = 2 \int_0^L \frac{\mu U_\infty}{\delta} \, b \, dx = 2 \int_2^L \frac{\mu U_\infty}{3.46} \sqrt{\frac{U_\infty}{v \, x}} \cdot b \, dx$$

$$= \frac{2 \mu U_\infty}{3.46} \sqrt{\frac{U_\infty}{v}} \, b \int_2^L \frac{1}{x^{(1/2)}} \, dx$$

$$= \frac{2\mu U_\infty b}{3.46} \sqrt{\frac{U_\infty}{\nu}} \left[2x^{\frac{1}{2}} \right]_o^L$$

$$= \frac{4\mu U_\infty b}{3.46} \sqrt{\frac{U_\infty L}{\nu}} = \frac{\mu U_\infty b}{3.46} \sqrt{\text{Re}_L}$$

$$\text{Re}_L = \frac{U_\infty L}{\nu} = \frac{1.2 \times 0.3}{1.1 \times 10^{-6}} = 3.27 \times 10^5$$

Therefore, $(\text{Re}_L)^{1/2} = 572$

Thus, $$F_D = \frac{4 \times 1.1 \times 10^{-3} \times 1.2 \times 2 \times 572}{3.46} = 1.745 \text{ N}$$

Example 9.6   Air is flowing over a thin flat plate which is 1 m long and 0.3 m wide. At the leading edge, the flow is assumed to be uniform and $U_\infty = 30$ m/s. The flow condition is independent of z (see Fig. 9.13). Using the control volume *abcd*, calculate the mass flow rate across the surface *ab*. Determine the magnitude and direction of the x component of force required to hold the plate stationary. The velocity profile at *bc* is given by

$$\frac{U}{U_\infty} = 2\left(\frac{y}{2}\right) - \left(\frac{y}{\delta}\right)^2$$

and $\delta = 4$ mm. Density of air $= 1.23$ kg/m^3 and $\nu = 1.5 \times 10^{-5}$ m^2/sec.
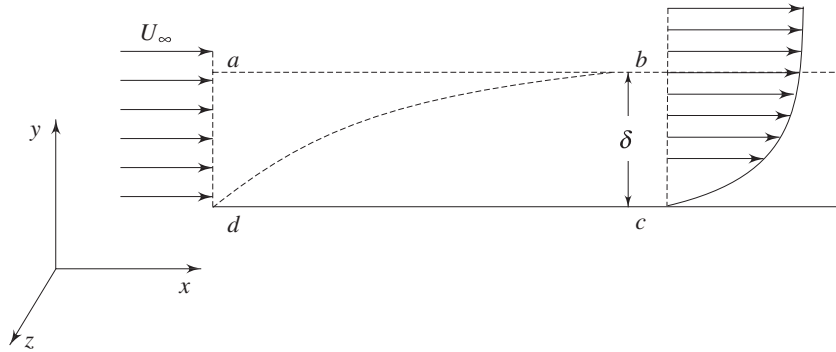

Fig. 9.13   Control volume of interest on flat plate

Solution   At *bc*, $\dfrac{U}{U_\infty} = 2\eta - \eta^2;\ \eta = y/\delta$

Steady state continuity equation is given by

$$\int_S \rho \vec{V} \cdot d\vec{A} = 0$$

or $$-\rho U_\infty b\delta + \int_0^\delta \rho u\, b\, dy + \dot{m}_{ab} = 0$$

but $\displaystyle\int_0^\delta \rho u \, b \, \mathrm{d}y = \rho U_\infty b\delta \int_0^1 (2\eta - \eta^2)\mathrm{d}\eta = \rho U_\infty b\delta \left[\eta^2 - \frac{\eta^3}{3}\right]_0^1 = \frac{2}{3}\rho U_\infty b\delta$

$$m_{ab} = \rho U_\infty b\delta - \frac{2}{3}\rho U_\infty b\delta = \frac{1}{3}\rho U_\infty b\delta$$

or $\qquad \dot{m}_{ab} = \dfrac{1}{3} \times 1.23 \times 30 \times 0.3 \times 0.004 = 0.0147 \text{ kg/s}$

Steady state momentum equation is given by

$$\int_{CS} u \rho \vec{V} \cdot \mathrm{d}\vec{A} = F_{sx}$$

or $\qquad R_x = u_{da}\{-\rho U_\infty b\delta\} + u_{ab} m_{ab} + \displaystyle\int_0^\delta u \rho \, ub \, \mathrm{d}y$

But, $\qquad u_{da} = u_{ab} = U_\infty, \quad$ and

$$\int_0^\delta u \rho u \, b \, \mathrm{d}y = \rho U_\infty^2 b\delta \int_0^1 (2\eta - \eta^2)^2 \mathrm{d}\eta$$

$$= \rho U_\infty^2 b\delta \left[\frac{4}{3}\eta^2 - \eta^4 + \frac{\eta^5}{3}\right]_0^1 = \frac{8}{15}\rho U_\infty^2 b\delta$$

Thus, $\qquad R_x = -\rho U_\infty^2 b\delta + \dfrac{1}{3}\rho U_\infty^2 b\delta + \dfrac{8}{15}\rho U_\infty^2 b\delta = -\dfrac{2}{15}\rho U_\infty^2 b\delta$

$$R_x = -\frac{2}{15}\{1.23 \times (3)^2 \times 0.3 \times 0.004\}$$

$$= -0.177 \text{ N (the force must be applied to the CV by the plate)}$$

Hence, $\qquad F_x = R_x = 0.177 \text{ N (to the left)}$

Exercises

9.1 **Choose the correct answer**
   (i) The laminar boundary layer thickness on a flat plate varies as
      (a) $x^{(-1/2)}$ (b) $x^{(4/5)}$ (c) $x^{(1/2)}$ (d) x^2
   (ii) The turbulent boundary layer thickness on a flat plate varies as

      (a) $x^{(+1/2)}$ (b) $x^{(4/5)}$ (c) $x^{(1/7)}$ (d) $x^{(6/7)}$
   (iii) The injection of air through a porous wall can control the boundary layer separation on the upper curved surface of an aerofoil wing. The injection of fluid also promotes turbulence. The final result is
      (a) increase in the skin friction and decrease in the form drag
      (b) increase in the form drag and decrease in the skin friction
      (c) decrease in both the skin friction and form drag
      (d) increase in both the skin friction and form drag

   (iv) In the entrance region of a pipe, the boundary layer grows and the inviscid core accelerates. This is accompanied by a
       (a) rise in pressure
       (b) constant pressure gradient
       (c) fall in pressure in the flow direction
       (d) pressure pulse

   (v) Flow separation is caused by
       (a) reduction of pressure to vapour pressure
       (b) a negative pressure gradient
       (c) a positive pressure gradient
       (d) the boundary layer thickness reducing to zero.

   (vi) At the point of separation
       (a) shear stress is zero
       (b) velocity is negative
       (c) pressure gradient is negative
       (d) shear stress is maximum

   (vii) Choose the expression for the momentum thickness of an incompressible boundary layer

       (a) $\dfrac{5.0\,x}{\sqrt{Re_x}}$ (b) $\displaystyle\int_0^\infty \left(1 - \dfrac{u}{U_\infty}\right) dy$

       (c) $\displaystyle\int_0^\infty \dfrac{u}{U_\infty}\left(1 - \dfrac{u}{U_\infty}\right) dy$ (d) $\displaystyle\int_0^\infty (u/U_\infty)\, dy$

   (viii) For cross flow over a circular cylinder at a Reynolds number $\left[Re = \dfrac{U_\infty D}{v}\right]$

       greater than 200, the wake is
       (a) at a pressure equal to the forward stagnation point
       (b) at a pressure lower than the forward stagnation point
       (c) the principal cause of skin friction
       (d) at a pressure higher than the forward stagnation point

9.2 Nikuradse, a well known student of Prandtl, obtained experimental data for laminar flow over a flat plate placed at zero angle of attack. His measurements suggest

$$\frac{u}{U} = a\left(\frac{y}{\delta}\right) + b\left(\frac{y}{\delta}\right)^3$$

where, y is the perpendicular distance from the plate. The local velocity is u. Evaluate a and b from physical boundary conditions. Obtain the expressions for the boundary layer thickness δ, displacement thickness δ^*, momentum thickness δ^{**} and the shear stress τ_w on the surface of the plate.

Ans. ($a = 3/2$, $b = -1/2$, $\delta/x = 4.64\,/(Re_x)^{0.5}$,
($\delta^*/x = 1.73\,(Re_x)^{0.5}$, $\tau_w = 0.323\,\mu\,(Re_x)^{0.5}\,U/x$

9.3 Given the choice between cos Ay and sin Ay as velocity profiles, which one would you prefer? To determine choice of the profile, find the displacement thickness, momentum thickness, wall shear stress and boundary layer thickness, from the momentum integral equation for flow over a flat plate.

Ans. (sin Ay, $\delta^* = 0.363\,\delta$, $\delta^{**} = 0.137\,\delta$, $\tau_w = \pi\mu\,U_\infty/2\delta$, $\delta/x = 4.8/(Re_x)^{0.5}$

9.4 For the laminar flow over a flat plate, the experiments confirm the velocity profile

$\dfrac{u}{U} = \dfrac{3}{2}\left(\dfrac{y}{\delta}\right) - \dfrac{1}{2}\left(\dfrac{y}{\delta}\right)^3$. For the turbulent flow over a flat plate, the

experimental observations over a range of Reynolds number suggest

$\dfrac{u}{U} = \left(\dfrac{y}{\delta}\right)^{1/7}$. Find the ratio of (δ^*/δ) for laminar and turbulent cases.

Ans. $(\delta^*/\delta)_{\text{lam}} = 3/8$, $(\delta^*/\delta)_{\text{tur}} = 1/8$

9.5 Assuming the velocity profile $\dfrac{u}{U_\infty} = \tanh \dfrac{y}{a(x)}$ for the boundary layer over a

flat plate at zero incidence, find the relations for $\delta, \delta^* \ \delta^{**}$ and τ_w. Check whether the profile satisfies all the boundary conditions.

Note: $a(x) \neq \delta(x)$ where δ is such that at $y = \delta, u = 0.99\ U_\infty$

Ans. $(\delta/x = 6.76/(\text{Re}_x)^{0.5}, \ \delta^*/x = 1.77/(\text{Re}_x)^{0.5} \ \delta^{**}/x = 0.783/(\text{Re}_x)^{0.5},$
$\tau_w = \mu\ U_\infty\ (\text{Re}_x)^{0.5}/2.553\ x$

9.6 An approximate expression for the velocity profile in a steady, 2–D, incompressible boundary layer is

$$\dfrac{u}{U_\infty} = 1 - e^{-\eta} + k\left(1 - e^{-\eta} - \sin \dfrac{\pi\eta}{6}\right), \quad \text{for } 0 \le \eta \le 3$$
$$= 1 - e^{-\eta} - ke^{-\eta}, \quad \text{for} \quad \eta \ge 3$$

where $\eta = y/\delta(x)$. Show that the profile satisfies the following boundary conditions

at $y = 0, u = 0$

at $y = \infty, u = U_\infty, \dfrac{\partial u}{\partial y} = \dfrac{\partial^2 u}{\partial y^2} = 0$

Also find out k from an appropriate boundary condition.

Ans. $(k = (\delta^2/v)\ (dU_\infty/dx) - 1)$

9.7 Water of kinematic viscosity $v = 1.02 \times 10^{-6}$ m^2/s is flowing steadily over a smooth flat plate at zero angle of attack with a velocity 1.6 m/s. The length of the plate is 0.3 m. Calculate (a) the thickness of boundary layer at 15 cm from the leading edge (b) the rate-of-growth of boundary layer at 15 cm from the leading edge, and (c) the total drag coefficient on one side of the plate. Assume a parabolic velocity profile.

Ans. $(\delta = 1.7$ mm, $d\delta/dx = 5.625 \times 10^{-3}, \overline{C}_f = 1.935 \times 10^{-3}$

9.8 Water flows between two parallel walls as shown in Fig. 9.14. The velocity is uniform at the entrance and core region. Beyond a distnace *Le* downstream from the entrance, the flow becomes fully developed so that the velocity varies over the entire width 2*h* of the channel. In the boundary layer region, velocity varies as

$u = U(x)\left(\dfrac{y}{\delta}\right)^2$ where $\delta = \alpha\ \sqrt{x}$; α being a constant. Determine the acceleration
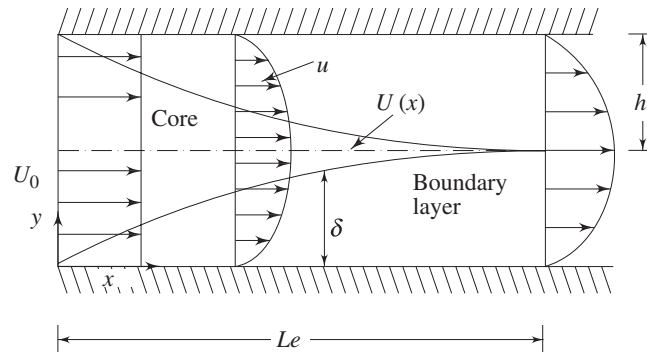
on the axis of symmetry for $0 \le x \le Le$.

Fig. 9.14 Development of boundary layers in a channel

Ans. $a = (U_0^2 / 3Le) (x/Le)^{-1/2}/(1 - (2/3)(x/Le)^{0.5})^3$

9.9 Consider the laminar boundary layer on a flat plate with uniform suction velocity V_0 as shwon in Fig. 9.15:
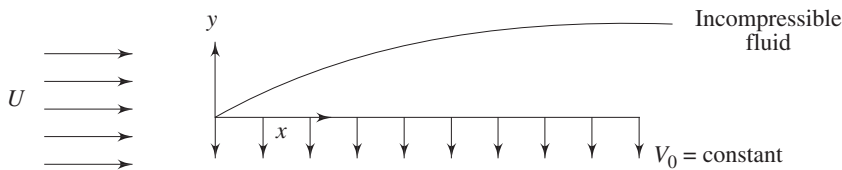

Fig. 9.15 Flow on a flat plate with uniform suction velocity

Far down the plate (large x), a fully-developed situation may be shown to exist in which the velocity distribution does not vary with x. Find the velocity distribution in this region, as well as the wall shear. The governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{and} \qquad u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + v\frac{\partial^2 u}{\partial y^2}$$

The boundary conditions are at $y = 0$, $u = 0$, $v = V_0$ and $u(\infty) = U$

Ans. $u = U\{1 - \exp(-V_0 y/v)\}$, $\tau_w = \rho U V_0$

9.10 Consider a laminar boundary layer on flat plate with a velocity profile given by

$$\frac{u}{U} = \frac{3}{2}\eta - \frac{1}{2}\eta^3, \qquad \text{where, } \eta = y/\delta$$

For this profile $\dfrac{\delta}{x} = \dfrac{4.64}{\text{Re}_x}$.

Determine an expression for the local skin friction coefficient in terms of distance and flow properties.

Ans. ($C_f = 0.647 \, (\text{Re}_x)^{0.5}$)

9.11 A low-speed wind-tunnel is provided with air supply upto a speed of 50 m/s at 20 °C. One needs to study the behaviour of boundary-layer over a flat plate kept

inside the wind-tunnel, up to a Reynolds number of $\text{Re}_x = 10^8$. What is the minimum plate length that should be used? At what distance from the leading edge would the transition occur if the critical Reynolds number is $\text{Re}_{x,\,c} = 3.5 \times 10^5$? At 25 °C, v of air is 15.7×10^{-6} m^2/s.

Ans. ($x_{\min} = 31.4$ m, $x_{\text{cr}} = 0.109$ m)

9.12 Perform a numerical solution (develop a FORTRAN code) using Eq. (9.24) and a Runge-Kutta method (as outlined in the text) which will iterate the Blasius equation from an initial guess $H(0) = 0.2$ and converge to the exact value of $H(0) = 0.4696$.

9.13 A liquid film of uniform thickness flows down the inner wall of vertical pipe, the thickness of the film being very small compared to the pipe radius. Show that, in the absence of any appreciable shear force on the free surface of the film, the volume flow of liquid per unit time, Q_1, is given by

$$Q_1 = \frac{2\pi rgt^3}{3v}$$

where, r is the pipe radius, g the gravitational acceleration, t is the thickness of the film and v is the kinematic viscosity of the liquid.

Show further that, if air flows up the pipe at such a rate that the free surface of the film remains at rest, then the volume flow of liquid per unit time, Q, is given by

$$Q \cong \frac{Q_1}{4}\left(1 - \frac{1}{\rho g}\frac{\mathrm{d}p}{\mathrm{d}x}\right).$$

where $\dfrac{\mathrm{d}p}{\mathrm{d}x}$ is the pressure drop along the pipe, ρ is the density of the liquid

and other symbols are as defined above.

9.14 The velocity distribution in the laminar boundary layer is of the form

$$\frac{u}{U_e} = F(\eta) + \lambda G(\eta)$$

where, $\quad F(\eta) = \dfrac{3}{2}\eta - \dfrac{\eta^3}{2};\; G(\eta) = \dfrac{\eta}{4} - \dfrac{\eta^2}{2} + \dfrac{\eta^3}{4};\; \eta = \dfrac{y}{\delta}$ and $\lambda = \dfrac{\delta^2}{v}\dfrac{\mathrm{d}Ue}{\mathrm{d}x}$

Find the value of λ when the flow is on the point of separating and show that then the displacement thickness will be half the boundary layer thickness.
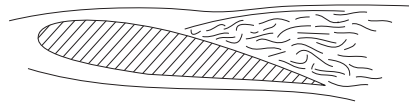
10 Turbulent Flow

10.1 INTRODUCTION

The turbulent motion is an irregular motion. The irregularity associated with turbulence is such that it can be described by the laws of probability and turbulent fluid motion can be considered as an irregular condition of flow in which various quantities (such as velocity components and pressure) show a random variation with time and space in such a way that the statistical average of those quantities can be quantitatively expressed.

An irregular motion is associated with random fluctuations. It is postulated that the fluctuations inherently come from disturbances (such as roughness of a solid surface) and they may be either dampened out due to viscous damping or may grow by drawing energy from the free stream. At a Reynolds number less than the critical, the kinetic energy of flow is not enough to sustain the random fluctuation against the viscous damping and in such cases laminar flow continues to exist. At somewhat higher Reynolds number than the critical Reynolds number, the kinetic energy of flow supports the growth of fluctuations and transition to turbulence takes place.

10.2 CHARACTERISTICS OF TURBULENT FLOW

In view of the preceding discussion, the most important characteristic of turbulent motion is the fact that velocity and pressure at a point fluctuate with time in a random manner (Fig. 10.1).
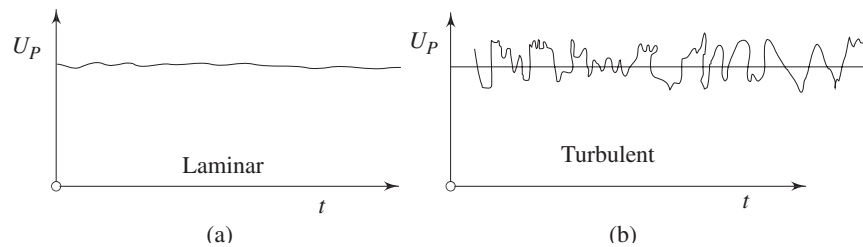
Fig. 10.1   Variation of *u* components of velocity for laminar and turbulent flows at a point *P*

The mixing in turbulent flow is more due to these fluctuations. As a result we can see more uniform velocity distributions in turbulent pipe flows as compared to laminar flows (see Fig. 10.2).
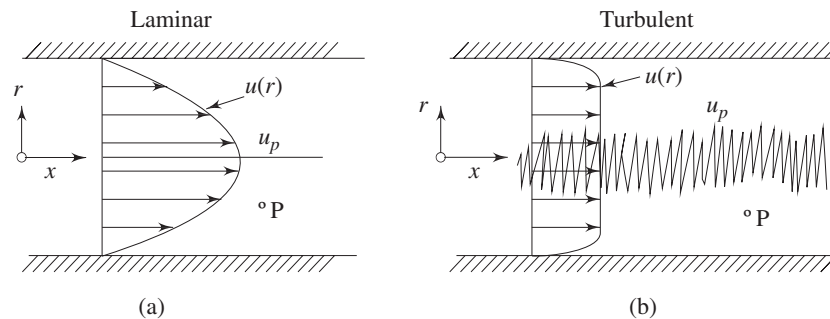

Fig. 10.2   Comparison of velocity profiles in a pipe for (a) laminar and (b) turbulent flows

Turbulence can be generated by frictional forces at the confining solid walls or by the flow of layers of fluids with different velocities over one another. The turbulence generated in these two ways are considered to be different. Turbulence generated and continuously affected by fixed walls is designated as *wall turbulence*, and turbulence generated by two adjacent layers of fluid in absence of walls is termed as *free turbulence*.

One of the effects of viscosity on turbulence is to make the flow more homogeneous and less dependent on direction. If the turbulence has the same structure quantitatively in all parts of the flow field, the turbulence is said to be *homogeneous*. Turbulence is called *isotropic* if its statistical features have no directional preference and perfect disorder persists. Its velocity fluctuations are independent of the axis of reference, i.e. invariant to axis rotation and reflection. Isotropic turbulence is by its definition always homogeneous. In such a situation, the gradient of the mean velocity does not exist. The mean velocity is either zero or constant throughout. However, when the mean velocity has a gradient, the turbulence is called *anisotropic*.

A little more discussion on homogeneous and isotropic turbulence is needed at this stage. The term homogeneous turbulence implies that the velocity fluctuations in the system are random. The average turbulent characteristics are independent of the position in the fluid, i.e., invariant to axis translation.
Consider the root mean square velocity fluctuations

$$u' = \sqrt{\overline{u^2}} , \; v' = \sqrt{\overline{v^2}} , \; w' = \sqrt{\overline{w^2}}$$

In homogeneous turbulence, the rms values of u', v' and w' can all be different, but each value must be constant over the entire turbulent field. It is also to be understood that even if the rms fluctuation of any component, say u's are constant over the entire field, the instantaneous values of u may differ from point to point at any instant.

In addition to its homogeneous nature, if the velocity fluctuations are independent of the axis of reference, i.e., invariant to axis rotation and reflection, the situation leads to isotropic turbulence, which by definition as mentioned earlier, is always homogeneous.

In isotropic turbulence fluctuations are independent of the direction of reference and

$$\sqrt{\overline{u^2}} = \sqrt{\overline{v^2}} = \sqrt{\overline{w^2}}$$

or $$u' = v' = w'$$

Again it is of relevance to say that even if the rms fluctuations at any point are same, their instantaneous values may differ from each other at any instant.

Turbulent flow is also diffusive. In general, turbulence brings about better mixing of a fluid and produces an additional diffusive effect. The term "eddy-diffusion" is often used to distinguish this effect from molecular diffusion. The effects caused by mixing are as if the viscosity is increased by a factor of 100 or more. At a large Reynolds number there exists a continuous transport of energy from the free stream to the large eddies. Then, from the large eddies smaller eddies are continuously formed. Near the wall smallest eddies dissipate energy and destroy themselves.

10.3 LAMINAR–TURBULENT TRANSITION

For turbulent flow over a flat plate, the boundary layer starts out as laminar flow at the leading edge and subsequently, the flow turns into transition flow and very shortly thereafter turns into turbulent flow. The turbulent boundary layer continues to grow in thickness, with a small region below it called a viscous sublayer. In this sublayer, the flow is well behaved, just as the laminar boundary layer (Fig. 10.3).

A careful observation further suggests that at a certain axial location, the laminar boundary layer tends to become unstable. Physically this means that the disturbances in the flow grow in amplitude at this location.
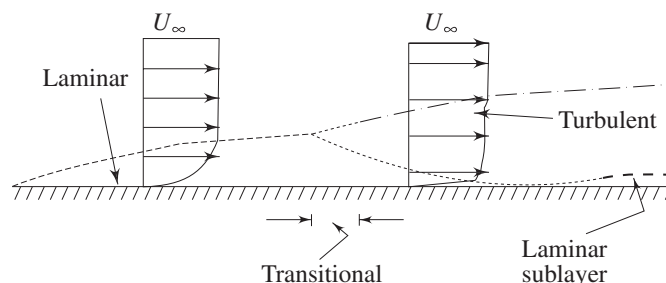
Fig. 10.3 Laminar-turbulent transition

Free stream turbulence, wall roughness and acoustic signals may be among the sources of such disturbances. Transition to turbulent flow is thus initiated with the instability in laminar flow. The possibility of instability in boundary layer was felt by Prandtl as early as 1912. The theoretical analysis of Tollmien and Schlichting showed that unstable waves could exist if the Reynolds number was 575. The Reynolds number was defined as

$$\mathrm{Re} = U_\infty \delta^*/\nu$$

where U_∞ is the free stream velocity and δ^* is the displacement thickness. Taylor developed an alternate theory, which assumed that the transition is caused by a momentary separation at the boundary layer associated with the free stream turbulence. In a pipe flow the initiation of turbulence is usually observed at Reynolds numbers ($U_\infty D/\nu$) in the range of 2000 to 2700. The development starts with a laminar profile, undergoes a transition, changes over to turbulent profile and then stays turbulent thereafter (Fig. 10.4). The length of development is of the order of 25 to 40 diameters of the pipe. The mechanisms related to the growth and the decay of turbulence in a flow field are indeed an advanced topic and beyond the scope of this text. However, the interested readers may like to refer to Tennekes and Lumley [1] and Hinze [2] for advanced knowledge.
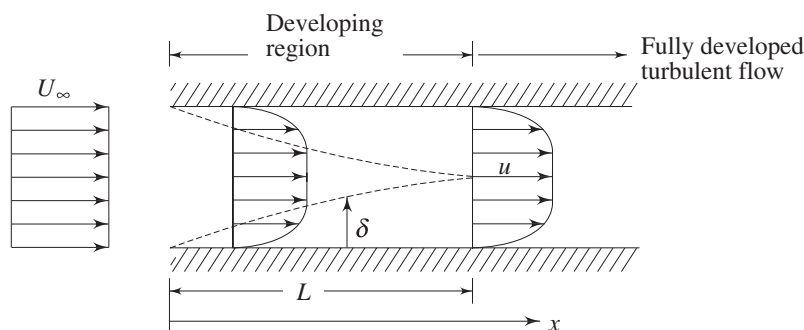

Fig. 10.4 Development of turbulent flow in a circular duct

10.4 CORRELATION FUNCTIONS

A statistical correlation can be applied to fluctuating velocity terms in turbulence. Turbulent motion is by definition eddying motion. Notwithstanding the circulation strength of the individual eddies, a high degree of correlation exists between the velocities at two points in space, if the distance between the points is smaller than the diameter of the eddy.

Consider a statistical property of a random variable (velocity) at two points separated by a distance r. An Eulerian correlation tensor (nine terms) at the two points can be defined by

$$\mathbf{Q} = \overline{\mathbf{u}(x)\mathbf{u}(x+r)}$$

In other words, the dependence between the two velocities at two points is measured by the correlations, i.e. the time averages of the products of the quantities measured at two points. The correlation of the u' components of the turbulent velocity of these two points is defined as

$$\overline{u'(x)u'(x+r)}$$

It is conventional to work with the non-dimensional form of the correlation, such as

$$R(r) = \frac{\overline{u'(x)u'(x+r)}}{\left(\overline{u'^2}(x)\overline{u'^2}(x+r)\right)^{1/2}}$$

A value of $R(r)$ of unity signifies a perfect correlation of the two quantities involved and their motion is in phase. Negative value of the correlation function implies that the time averages of the velocities in the two correlated points have different signs. Figure 10.5 shows typical variations of the correlation R with increasing separation r.

To describe the evolution of a fluctuating function $u'(t)$, we need to know the manner in which the value of u' at different times are related. For this purpose a correlation function

$$R(\tau) = \overline{u'(t)u'(t+\tau)}\big/u'^2$$

between the values of u' at different times has been chosen. This is called autocorrelation function.

The correlation studies reveal that the turbulent motion is composed of eddies which are convected by the mean motion. The eddies have a wide range variation in their size. The size of the large eddies is comparable with the dimensions of the neighbouring objects or the dimensions of the flow passage. The size of the smallest eddies can be of the order of 1 mm or less. However, the smallest eddies are much larger than the molecular mean free paths and the turbulent motion obeys the principles of continuum mechanics.
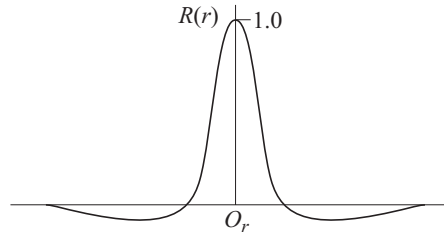
Fig. 10.5 Variation of R with the distance of separation r

10.5 MEAN MOTION AND FLUCTUATIONS

In 1883, O. Reynolds conducted experiments with pipe flow by feeding into the stream a thin thread of liquid dye. For low Reynolds numbers, the dye traced a straight line and did not disperse. With increasing velocity, the dye thread got mixed in all directions and the flowing fluid appeared to be uniformly coloured in the downstream. It was conjectured that on the main motion in the direction of the axis, there existed a superimposed motion all along the main motion at right angles to it. The superimposed motion causes exchange of momentum in transverse direction and the velocity distribution over the cross-section is more uniform than in laminar flow. This description of turbulent flow which consists of superimposed streaming and fluctuating (eddying) motion is referred to as Reynolds decomposition of turbulent flow.

Here, we shall discuss different descriptions of mean motion. Generally, for Eulerian velocity u, the following two methods of averaging could be obtained.

(i) Time average for a stationary turbulence

$$\bar{u}^t(x_0) = \lim_{t_1 \to \infty} \frac{1}{2t_1} \int_{-t_1}^{t_1} u(x_0, t)\, dt$$

(ii) Space average for a homogeneous turbulence

$$\bar{u}^s(t_0) = \lim_{x \to \infty} \frac{1}{2x} \int_{-x}^{x} u(x, t_0)\, dx$$

For a stationary and homogeneous turbulence, it is assumed that the two averages lead to the same result: $\bar{u}^t = \bar{u}^s$ and the assumption is known as the *ergodic hypothesis.*

In our analysis, average of any quantity will be evaluated as a *time average.* We take t_1 a finite interval. This interval must be larger than the time scale of turbulence. Needless to say that it must be small compared with the period t_2 of any slow variation (such as periodicity of the mean flow) in the flow field that we do not consider to be *chaotic* or *turbulent.*

Thus, for a parallel flow, it can be written that the axial velocity component is

$$u(y, t) = \bar{u}(y) + u'(\Gamma, t) \tag{10.1}$$

As such, the time mean component $\bar{u}\,(y)$ determines whether the turbulent motion is steady or not. Let us look at two different turbulent motions described in Fig. 10.6 (a) and (b). The symbol Γ signifies any of the space variables.

While the motion described by Fig. 10.6 (a) is for a turbulent flow with steady mean velocity the Fig. 10.6 (b) shows an example of turbulent flow with unsteady mean velocity. The time period of the high frequency fluctuating component is t_1 whereas the time period for the unsteady mean motion is t_2 and for obvious reason $t_2 >> t_1$. Even if the bulk motion is parallel, the fluctuation u' being random varies in all directions. Now let us look at the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Invoking Eq. (10.1) in the above expression, we get

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{10.2}$$
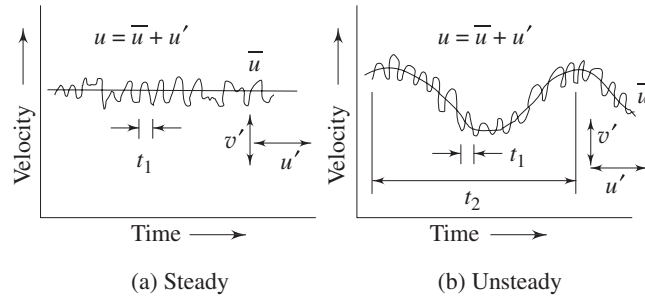
(a) Steady   (b) Unsteady

Fig. 10.6   Steady and unsteady mean motions in a turbulent flow

Since $\dfrac{\partial u'}{\partial x} \neq 0$, Eq. (10.2) depicts that y and z components of velocity exist even for the parallel flow if the flow is turbulent. We can write

$$\left.\begin{array}{l} u(y,\,t) = \bar{u}\,(y) + u'\,(\Gamma,\,t) \\ v = \ 0 + v'\,(\Gamma,\,t) \\ w = 0 + w'\,(\Gamma,\,t) \end{array}\right\} \tag{10.3}$$

However, the fluctuating components do not bring about the bulk displacement of a fluid element. The instantaneous displacement is $u'\,dt$, and that is not responsible for the bulk motion. We can conclude from the above

$$\int_{-T}^{T} u'\,dt = 0, \qquad (t_1 < T \le t_2)$$

Due to the interaction of fluctuating components, macroscopic momentum transport takes place. Therefore, interaction effect between two fluctuating components over a long period is non-zero and this can be expressed as

$$\int_{-T}^{T} u' \, v' \, \mathrm{d}t \neq 0$$

We take time average of these two integrals and write

$$\overline{u'} = \frac{1}{2T} \int_{-T}^{T} u' \, \mathrm{d}t = 0 \tag{10.4a}$$

and

$$\overline{u'\,v'} = \frac{1}{2T} \int_{-T}^{T} u' \, v' \, \mathrm{d}t \neq 0 \tag{10.4b}$$

Now, we can make a general statement with any two fluctuating parameters, say, with f' and g' as

$$\overline{f'} = \overline{g'} = 0 \tag{10.5a}$$

$$\overline{f'g'} \neq 0 \tag{10.5b}$$

The time averages of the spatial gradients of the fluctuating components also follow the same laws, and they can be written as

$$\left.\begin{array}{l} \dfrac{\overline{\partial f'}}{\partial s} = \dfrac{\overline{\partial^2 f'}}{\partial s^2} = 0 \\[3mm] \dfrac{\overline{\partial (f'g')}}{\partial s} \neq 0 \end{array}\right\} \tag{10.6}$$

and

The *intensity of turbulence* or *degree of turbulence* in a flow is described by the relative magnitude of the root mean square value of the fluctuating components with respect to the time averaged main velocity. The mathematical expression is given by

$$I = \sqrt{\frac{1}{3}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})} \Big/ U_\infty \tag{10.7a}$$

The degree of turbulence in a wind tunnel can be brought down by introducing screens of fine mesh at the bell mouth entry. In general, at a certain distance from the screens, the turbulence in a wind tunnel becomes isotropic, i.e. the mean oscillation in the three components are equal,

$$\overline{u'^2} = \overline{v'^2} = \overline{w'^2}$$

In this case, it is sufficient to consider the oscillation u' in the direction of flow and to put

$$I = \sqrt{\overline{u'^2}} \Big/ U_\infty \tag{10.7b}$$

This simpler definition of turbulence intensity is often used in practice even in cases when turbulence is not isotropic.

Following Reynolds decomposition, it is suggested to separate the motion into a mean motion and a fluctuating or eddying motion. Denoting the time average of

the u component of velocity by $\bar{u}$ and fluctuating component as u', we can write down the following,

$$u = \bar{u} + u', \quad v = \bar{v} + v', \quad w = \bar{w} + w', \quad p = \bar{p} + p' \tag{10.8}$$

By definition, the time averages of all quantities describing fluctuations are equal to zero.

$$\overline{u'} = 0, \quad \overline{v'} = 0, \quad \overline{w'} = 0, \quad \overline{p'} = 0 \tag{10.9}$$

The fluctuations u', v', and w' influence the mean motion $\bar{u}$, $\bar{v}$ and $\bar{w}$ in such a way that the mean motion exhibits an apparent increase in the resistance to deformation. In other words, the effect of fluctuations is an apparent increase in viscosity or macroscopic momentum diffusivity.

We shall state some rules of mean time-averages here. If f and g are two dependent variables and if s denotes any one of the independent variables x, y, z, t, then

$$\bar{\bar{f}} = \bar{f}; \quad \overline{f + g} = \bar{f} + \bar{g}; \quad \overline{\bar{f} \cdot g} = \bar{f} \cdot \bar{g};$$

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \bar{f}}{\partial s}; \quad \int \overline{f \, ds} = \int \bar{f} \, ds \tag{10.10}$$

10.6 DERIVATION OF GOVERNING EQUATIONS FOR TURBULENT FLOW

For incompressible flows, the Navier–Stokes equations can be rearranged in the form

$$\rho \left[\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} \right] = -\frac{\partial p}{\partial x} + \mu \nabla^2 u \tag{10.11a}$$

$$\rho \left[\frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (v^2)}{\partial y} + \frac{\partial (vw)}{\partial z} \right] = -\frac{\partial p}{\partial y} + \mu \nabla^2 v \tag{10.11b}$$

$$\rho \left[\frac{\partial w}{\partial t} + \frac{\partial (uw)}{\partial x} + \frac{\partial (vw)}{\partial y} + \frac{\partial (w^2)}{\partial z} \right] = -\frac{\partial p}{\partial z} + \mu \nabla^2 w \tag{10.11c}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{10.12}$$

Let us express the velocity components and pressure in terms of time-mean values and corresponding fluctuations. In continuity equation, this substitution and subsequent time averaging will lead to

$$\frac{\overline{\partial (\bar{u} + u')}}{\partial x} + \frac{\overline{\partial (\bar{v} + v')}}{\partial y} + \frac{\overline{\partial (\bar{w} + w')}}{\partial z} = 0$$

or

$$\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} \right) + \left(\frac{\overline{\partial u'}}{\partial x} + \frac{\overline{\partial v'}}{\partial y} + \frac{\overline{\partial w'}}{\partial z} \right) = 0$$

Since $\quad \dfrac{\partial \overline{u'}}{\partial x} = \dfrac{\partial \overline{v'}}{\partial y} = \dfrac{\partial \overline{w'}}{\partial z} = 0 \qquad$ [From Eq. (10.6)]

We can write

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0 \tag{10.13a}$$

From Eqs (10.13a) and (10.12), we obtain

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{10.13b}$$

It is evident that the time-averaged velocity components and the fluctuating velocity components, each satisfy the continuity equation for incompressible flow. Let us imagine a two-dimensional flow in which the turbulent components are independent of the *z*-direction. Eventually, Eq. (10.13b) tends to

$$\frac{\partial u'}{\partial x} = - \frac{\partial v'}{\partial y} \tag{10.14}$$

On the basis of condition (10.14), it is postulated that if at an instant there is an increase in u' in the *x*-direction, it will be followed by an increase in v' in the negative *y*-direction. In other words, $\overline{u'v'}$ is non-zero and negative. This is another important consideration within the framework of mean-motion description of turbulent flows.

Invoking the concepts of (10.8) into the equations of motion (10.11a, b, c), we obtain expressions in terms of mean and fluctuating components. Now, forming time averages and considering the rules of (10.10) we discern the following. The terms which are linear, such as $\dfrac{\partial u'}{\partial t}$ and $\dfrac{\partial^2 u'}{\partial x^2}$ vanish when they are averaged [from (10.6)]. The same is true for the mixed terms like $\overline{u} \cdot u'$, or $\overline{u} \cdot v'$, but the quadratic terms in the fluctuating components remain in the equations. After averaging, they form $\overline{u'^2}$, $\overline{u'v'}$ etc.

For example, if we perform the aforesaid exercise on the *x* momentum equation, we shall obtain

$$\rho \left[\frac{\overline{\partial(\overline{u}+u')}}{\mathrm{d}t} + \frac{\overline{\partial(\overline{u}+u')^2}}{\partial x} + \frac{\overline{\partial(\overline{u}+u')(\overline{v}+v')}}{\partial y} + \frac{\overline{\partial(\overline{u}+u')(\overline{w}+w')}}{\partial z} \right]$$

$$= - \frac{\overline{\partial(\overline{p}+p')}}{\partial x} + \mu \left[\frac{\overline{\partial^2(\overline{u}+u')}}{\partial x^2} + \frac{\overline{\partial^2(\overline{u}+u')}}{\partial y^2} + \frac{\overline{\partial^2(\overline{u}+u')}}{\partial z^2} \right]$$

or $\quad \rho \left\{ \dfrac{\partial \overline{u}}{\partial t}^{\!\!\!0} + \dfrac{\partial \overline{u'}}{\partial t}^{\!\!\!0} + \dfrac{\partial(\overline{u}^2 + \overline{u'^2})}{\partial x} + \dfrac{\partial(\overline{u}\cdot\overline{v} + \overline{u'v'})}{\partial y} + \dfrac{\partial(\overline{u}\cdot\overline{w} + \overline{u'w'})}{\partial z} \right\}$

$$= -\frac{\partial \bar{p}}{\partial x} - \frac{\partial \bar{p}}{\partial x}^{0} + \mu\left[\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2} + \left(\frac{\partial^2 \bar{u}}{\partial x^2}^{0} + \frac{\partial^2 \bar{u}}{\partial y^2}^{0} + \frac{\partial^2 \bar{u}}{\partial z^2}^{0}\right)\right]$$

or

$$\rho\left\{\frac{\partial(\bar{u}^2)}{\partial x} + \frac{\partial(\bar{u}\cdot\bar{v})}{\partial y} + \frac{\partial(\bar{u}\cdot\bar{w})}{\partial z}\right\} = -\frac{\partial \bar{p}}{\partial x} + \mu\nabla^2 \bar{u}$$

$$-\rho\left[\frac{\partial}{\partial x}\overline{u'^2} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'}\right]$$

Introducing simplifications arising out of continuity Eq. (10.13a), we shall obtain

$$\rho\left[\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}\right] = -\frac{\partial \bar{p}}{\partial x} + \mu\nabla^2 \bar{u}$$

$$-\rho\left[\frac{\partial}{\partial x}\overline{u'^2} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'}\right]$$

Performing a similar treatment on y and z momentum equations, finally we obtain the momentum equations in the form

$$\rho\left[\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}\right] = -\frac{\partial \bar{p}}{\partial x} + \mu\nabla^2 \bar{u}$$

$$-\rho\left[\frac{\partial}{\partial x}\overline{u'^2} + \frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\overline{u'w'}\right] \tag{10.15a}$$

$$\rho\left[\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \bar{w}\frac{\partial \bar{v}}{\partial z}\right] = -\frac{\partial \bar{p}}{\partial y} + \mu\nabla^2 \bar{v}$$

$$-\rho\left[\frac{\partial}{\partial x}\overline{u'v'} + \frac{\partial}{\partial y}\overline{v'^2} + \frac{\partial}{\partial z}\overline{v'w'}\right] \tag{10.15b}$$

$$\rho\left[\bar{u}\frac{\partial \bar{w}}{\partial x} + \bar{v}\frac{\partial \bar{w}}{\partial y} + \bar{w}\frac{\partial \bar{w}}{\partial z}\right] = -\frac{\partial \bar{p}}{\partial z} + \mu\nabla^2 \bar{w}$$

$$-\rho\left[\frac{\partial}{\partial x}\overline{u'w'} + \frac{\partial}{\partial y}\overline{v'w'} + \frac{\partial}{\partial z}\overline{w'^2}\right] \tag{10.15c}$$

The left hand side of Eqs (10.15a)–(10.15c) are essentially similar to the steady-state Navier–Stokes equations if the velocity components u, v and w are replaced by $\bar{u}$, $\bar{v}$ and $\bar{w}$. The same argument holds good for the first two terms on the right hand side of Eqs (10.15a)–(10.15c). However, the equations contain some additional terms which depend on turbulent fluctuations of the stream. These additional terms can be interpreted as components of a stress tensor. Now, the resultant surface force per unit area due to these terms may be considered as

$$\rho\left[\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}\right] = -\frac{\partial \bar{p}}{\partial x} + \mu\nabla^2 \bar{u}$$

$$+\left[\frac{\partial}{\partial x}\sigma'_{xx}+\frac{\partial}{\partial y}\tau'_{yx}+\frac{\partial}{\partial z}\tau'_{zx}\right] \tag{10.16a}$$

$$\rho\left[\bar{u}\frac{\partial\overline{\varpi}}{\partial x}+\overline{\varpi}\frac{\partial\overline{\varpi}}{\partial y}+\bar{w}\frac{\partial\overline{\varpi}}{\partial z}\right]=-\frac{\partial\bar{p}}{\partial y}+\mu\nabla^2\,\overline{\varpi}$$

$$+\left[\frac{\partial}{\partial x}\tau'_{xy}+\frac{\partial}{\partial y}\sigma'_{yy}+\frac{\partial}{\partial z}\tau'_{zy}\right] \tag{10.16b}$$

$$\rho\left[\bar{u}\frac{\partial\bar{w}}{\partial x}+\bar{v}\frac{\partial\bar{w}}{\partial y}+\bar{w}\frac{\partial\bar{w}}{\partial z}\right]=-\frac{\partial\bar{p}}{\partial z}+\mu\nabla^2\,\bar{w}$$

$$+\left[\frac{\partial}{\partial x}\tau'_{xz}+\frac{\partial}{\partial y}\tau'_{yz}+\frac{\partial}{\partial z}\sigma'_{zz}\right] \tag{10.16c}$$

Comparing Eqs (10.15) and (10.16), we can write

$$\begin{bmatrix}\sigma'_{xx}&\tau'_{xy}&\tau'_{xz}\\\tau'_{xy}&\sigma'_{yy}&\tau'_{yz}\\\tau'_{xz}&\tau'_{yz}&\sigma'_{zz}\end{bmatrix}=-\rho\begin{bmatrix}\overline{u'^2}&\overline{u'v'}&\overline{u'w'}\\\overline{u'v'}&\overline{v'^2}&\overline{v'w'}\\\overline{u'w'}&\overline{v'w'}&\overline{w'^2}\end{bmatrix} \tag{10.17}$$

It can be said that the mean velocity components of turbulent flow satisfy the same Navier–Stokes equations of laminar flow. However, for the turbulent flow, the laminar stresses must be increased by additional stresses which are given by the stress tensor (10.17). These additional stresses are known as apparent stresses of turbulent flow or *Reynolds stresses*. Since turbulence is considered as eddying motion and the aforesaid additional stresses are added to the viscous stresses due to mean motion in order to explain the complete stress field, it is often said that the apparent stresses are caused by eddy viscosity. The total stresses are now

$$\sigma_{xx}=-\bar{p}+2\mu\frac{\partial\bar{u}}{\partial x}-\overline{\rho u'^2}$$
$$\tau_{xy}=\mu\left(\frac{\partial\bar{u}}{\partial y}+\frac{\partial\bar{v}}{\partial x}\right)-\overline{\rho u'v'} \tag{10.18}$$

and so on. The apparent stresses are much larger than the viscous components, and the viscous stresses can even be dropped in many actual calculations.

10.7 TURBULENT BOUNDARY LAYER EQUATIONS

For a two-dimensional flow ($w = 0$) over a flat plate, the thickness of turbulent boundary layer is assumed to be much smaller than the axial length and the order of magnitude analysis (refer to Chapter-9) may be applied. As a consequence, the following inferences are drawn:

(a) $\dfrac{\partial\bar{p}}{\partial y}=0,$ (b) $\dfrac{\partial\bar{p}}{\partial x}=\dfrac{\mathrm{d}\bar{p}}{\mathrm{d}x}$

(c) $\dfrac{\partial^2 \overline{u}}{\partial x^2} << \dfrac{\partial^2 \overline{u}}{\partial y^2}$, and

(d) $\dfrac{\partial}{\partial x}\left(-\overline{\rho u'^2}\right) << \dfrac{\partial}{\partial y}\left(-\overline{\rho u' v'}\right)$

The turbulent boundary layer equation together with the equation of continuity becomes

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \tag{10.19}$$

$$\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{v}\,\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}\overline{p}}{\mathrm{d}x} + \frac{\partial}{\partial y}\left[v\frac{\partial \overline{u}}{\partial y} - \overline{u'v'}\right] \tag{10.20}$$

A comparison of Eq. (10.20) with laminar boundary layer Eq. (9.10) depicts. that: u, v and p are replaced by the time average values $\overline{u}$, $\overline{v}$ and $\overline{p}$, and laminar viscous force per unit volume $\dfrac{\partial(\tau_l)}{\partial y}$ is replaced by $\dfrac{\partial}{\partial y}$ $(\tau_1 + \tau_t)$, where $\tau_1 = \mu\dfrac{\partial \overline{u}}{\partial y}$ is the laminar shear stress and $\tau_t = -\rho u'\,v'$ is the turbulent stress.

10.8 BOUNDARY CONDITIONS

All the components of apparent stresses vanish at the solid walls and only stresses which act near the wall are the viscous stresses of laminar flow. The boundary conditions, to be satisfied by the mean velocity components, are similar to laminar flow. A very thin layer next to the wall behaves like a near wall region of the laminar flow. This layer is known as *laminar sublayer* and its velocities are such that the viscous forces dominate over the inertia forces. No turbulence exists in it (see Fig. 10.7). For a developed turbulent flow over a flat plate, in the near wall region, inertial effects are insignificant, and we can write from Eq. (10.20),

$$v\,\frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial \overline{(u'\,v')}}{\partial y} = 0$$
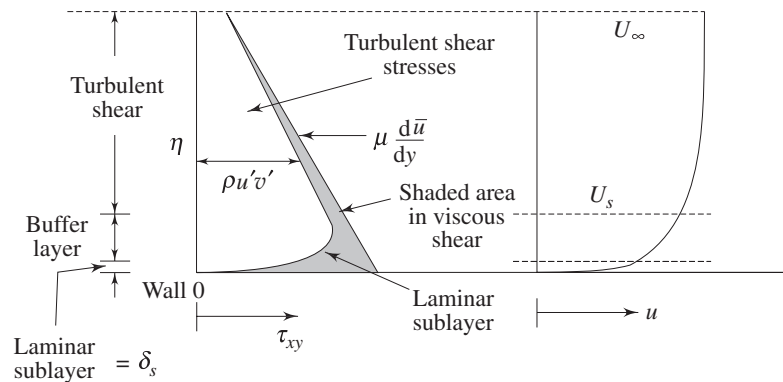

Fig. 10.7 Different zones of a turbulent flow past a wall

which can be integrated as, $\dfrac{\nu \partial \bar{u}}{\partial y} - \overline{u' v'} = \text{constant}$

Again, as we know that the fluctuating components, do not exist near the wall, the shear stress on the wall is purely viscous and it follows

$$\nu \frac{\partial \bar{u}}{\partial y}\bigg|_{y=0} = \frac{\tau_w}{\rho}$$

However, the wall shear stress in the vicinity of the laminar sublayer is estimated as

$$\tau_w = \mu \left[\frac{U_s - 0}{\delta_s - 0} \right] = \mu \frac{U_s}{\delta_s} \tag{10.21a}$$

where U_s is the fluid velocity at the edge of the sublayer. The flow in the sublayer is specified by a velocity scale (characteristic of this region). We define the friction velocity,

$$u_\tau = \left[\frac{\tau_w}{\rho} \right]^{1/2} \tag{10.21b}$$

as our velocity scale. Once u_τ is specified, the structure of the sublayer is specified. It has been confirmed experimentally that the turbulent intensity distributions are scaled with u_τ. For example, maximum value of the $\overline{u'^2}$ is always about $8u_\tau^2$. The relationship between u_τ and the U_s can be determined from Eqs (10.21a) and (10.21b) as

$$u_\tau^2 = \nu \frac{U_s}{\delta_s}$$

Let us assume $\quad U_s = \overline{C} U_\infty$. Now we can write

$$u_\tau^2 = \overline{C} \nu \frac{U_\infty}{\delta_s} \text{ where } \overline{C} \text{ is a proportionality constant} \tag{10.22a}$$

or $\qquad \dfrac{\delta_s u_\tau}{\nu} = \overline{C} \left[\dfrac{U_\infty}{u_\tau} \right] \tag{10.22b}$

Hence, a non-dimensional coordinate may be defined as, $\eta = \dfrac{y u_\tau}{\nu}$ which will help us estimating different zones in a turbulent flow. The thickness of laminar sublayer or viscous sublayer is considered to be $\eta \approx 5$. Turbulent effect starts in the zone of $\eta > 5$ and in a zone of $5 < \eta < 70$, laminar and turbulent motions coexist. This domain is termed as buffer zone. Turbulent effects far outweigh the laminar effect in the zone beyond $\eta = 70$ and this regime is termed as turbulent core.

For flow over a flat plate, the turbulent sheat stress $(-\rho \overline{u' v'})$ is constant throughout in the y direction and this becomes equal to τ_w at the wall. In the event

of flow through a channel, the turbulent shear stress $(-\rho\overline{u'v'})$ varies with y and it is possible to write

$$\frac{\tau_t}{\tau_w} = \frac{\zeta}{h} \tag{10.22c}$$

where the channel is assumed to have a height $2h$ and ζ is the distance measured from the centreline of the channel $(= h - y)$. Figure 10.7 explains such variation of turbulent stress.

10.9 SHEAR STRESS MODELS

In analogy with the coefficient of viscosity for laminar flow, J. Boussinesq introduced a mixing coefficient μ_τ for the Reynolds stress term by invoking

$$\tau_t = -\overline{\rho u'v'} = \mu_t \frac{\partial \overline{u}}{\partial y}$$

Now the expressions for shearing stresses are written as

$$\tau_l = \rho v \frac{\partial u}{\partial y}, \tau_t = \mu_t \frac{\partial \overline{u}}{\partial y} = \rho v_t \frac{\partial \overline{u}}{\partial y}$$

such that the equation

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\cdot\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left\{ v\frac{\partial \overline{u}}{\partial y} - \overline{u'v'} \right\}$$

may be written as

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{1}{\rho}\cdot\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left\{ (v + v_t)\frac{\partial \overline{u}}{\partial y} \right\} \tag{10.23}$$

The term v_t is known as *eddy viscosity* and the model is known as *eddy viscosity* model. The difficulty in using Eq. (10.23) can be discussed herein. The value of v_t is not known. The term v is a property of the fluid whereas v_t is attributed to random fluctuations and is not a property of the fluid. However, it is necessary to find out empirical relations between v_t and the mean velocity. We shall discuss one such well known relation between the aforesaid apparent or eddy viscosity and the mean velocity components in the following subsection.

10.9.1 Prandtl's Mixing Length Hypothesis

Let us consider a fully developed turbulent boundary layer (Fig. 10.3). The streamwise mean velocity varies only from streamline to streamline. The main flow direction is assumed parallel to the x-axis (Fig. 10.8).

The time average components of velocity are given by $\overline{u} = \overline{u}(y)$, $\overline{v} = 0$, $\overline{w} = 0$. The fluctuating component of transverse velocity v' transports mass and momentum across a plane at y_1 from the wall. The shear stress due to the fluctuation is

given by

$$\tau_t = -\rho\,\overline{u'v'} = \mu_t\,\frac{\partial \overline{u}}{\partial y} \tag{10.24}$$

A lump of fluid, which comes to the layer y_1 from a layer $(y_1 - l)$ has a positive value of v'. If the lump of fluid retains its original momentum then its velocity at its current location y_1 is smaller than the velocity prevailing there. The difference in velocities is then

$$\Delta u_1 = \overline{u}\ (y_1) - \overline{u}\ (y_1 - l) \approx l\left(\frac{\partial \overline{u}}{\partial y}\right)_{y_1} \tag{10.25}$$
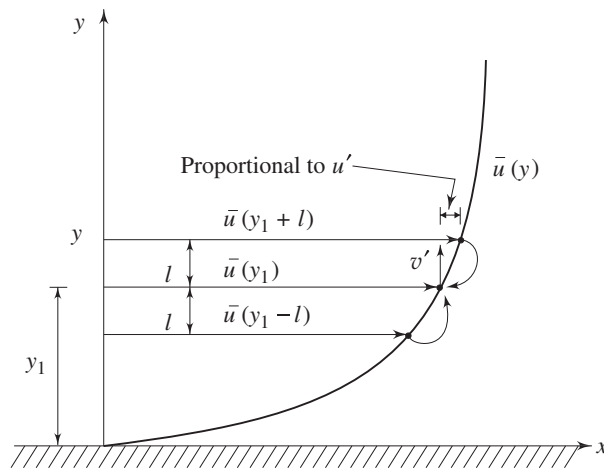

Fig. 10.8 One-dimensional parallel flow and Prandtl's mixing length hypothesis

The above expression is obtained by expanding the function $\overline{u}\ (y_1 - l)$ in a Taylor series and neglecting all higher order terms and higher order derivatives. As it is said, l is a small length scale known as Prandtl's mixing length. Prandtl proposed that the transverse displacement of any fluid particle is, on an average, 'l'. Let us consider another lump of fluid with a negative value of v'. This is arriving at y_1 from $(y_1 + l)$. If this lump retains its original momentum, its mean velocity at the current lamina y_1 will be somewhat more than the original mean velocity of y_1. This difference is given by

$$\Delta u_2 = \overline{u}\ (y_1 + l) - \overline{u}\ (y_1) \approx l\left(\frac{\partial \overline{u}}{\partial y}\right)_{y_1} \tag{10.26}$$

The velocity differences caused by the transverse motion can be regarded as the turbulent velocity components at y_1. We calculate the time average of the absolute value of this fluctuation as

$$\overline{|u'|} = \frac{1}{2}\left(|\Delta u_1| + |\Delta u_2|\right) = l\left|\left(\frac{\partial \overline{u}}{\partial y}\right)\right|_{y_1} \tag{10.27}$$

Suppose these two lumps of fluid meet at a layer y_1. The lumps will collide with a velocity $2u'$ and diverge. This proposes the possible existence of transverse velocity component in both directions with respect to the layer at y_1. Now, suppose that the two lumps move away in a reverse order from the layer y_1 with a velocity $2u'$. The empty space will be filled from the surrounding fluid creating transverse velocity components which will again collide at y_1. Keeping in mind this argument and the physical explanation accompanying Eqs (10.14), we may state that

$$\overline{|v'|} \sim \overline{|u'|}$$

or
$$\overline{|v'|} = \text{const } \overline{|u'|} = (\text{const}) \, l \left| \frac{\partial \bar{u}}{\partial y} \right|$$

along with the condition that the moment at which u' is positive, v' is more likely to be negative and conversely when u' is negative. Possibly, we can write at this stage

$$\overline{u' v'} = - C_1 \, \overline{|u'|} \, \overline{|v'|}$$

or
$$\overline{u' v'} = - C_2 \, l^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2 \tag{10.28}$$

where C_1 and C_2 are different proportionality constants. However, the constant C_2 can now be included in still unknown mixing length and Eq. (10.28) may be rewritten as

$$\overline{u' v'} = - l^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2$$

For the expression of turbulent shearing stress τ_t, we may write

$$\tau_t = - \rho \, \overline{u' v'} = \rho \, l^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2 \tag{10.29}$$

After comparing this expression with the eddy viscosity concept and Eq. (10.24), we may arrive at a more precise definition,

$$\tau_t = \rho \, l^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \left(\frac{\partial \bar{u}}{\partial y} \right) = \mu_t \frac{\partial \bar{u}}{\partial y} \tag{10.30a}$$

where the apparent viscosity may be expressed as

$$\mu_t = \rho \, l^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \tag{10.30b}$$

and the apparent kinematic viscosity is given by

$$v_t = l^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \tag{10.30c}$$

The decision of expressing one of the velocity gradients of Eq. (10.29) in terms of its modulus as $\left|\dfrac{\partial \bar{u}}{\partial y}\right|$ was made in order to assign a sign to τ_t according to the sign of $\dfrac{\partial \bar{u}}{\partial y}$. It may be mentioned that the apparent viscosity and consequently, the mixing length are not properties of fluid. They are dependent on turbulent fluctuation. However, our problem is still not resolved. How to determine the value of "*l*" the mixing length? Several correlations, using experimental results for τ_t have been proposed to determine *l*.

However, so far the most widely used value of mixing length in the regime of isotropic turbulence is given by

$$l = \chi y \tag{10.31}$$

where *y* is the distance from the wall and χ is known as von Karman constant (≈ 0.4).

10.10 UNIVERSAL VELOCITY DISTRIBUTION LAW AND FRICTION FACTOR IN DUCT FLOWS FOR VERY LARGE REYNOLDS NUMBERS

For flows in a rectangular channel at very large Reynolds numbers the laminar sublayer can practically be ignored. The channel may be assumed to have a width $2h$ and the *x* axis will be placed along the bottom wall of the channel. We shall consider a turbulent stream along a smooth flat wall in such a duct and denote the distance from the bottom wall by *y*, while $u(y)$ will signify the velocity. In the neighbourhood of the wall, we shall apply

$$l = \chi y$$

According to Prandtl's assumption, the turbulent shearing stress will be

$$\tau_t = \rho \chi^2 y^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 \tag{10.32}$$

At this point, Prandtl introduced an additional assumption which like a plane Couette flow takes a constant shearing stress throughout, i.e

$$\tau_t = \tau_w \tag{10.33}$$

where τ_w denotes the shearing stress at the wall. Invoking once more the friction velocity $u_\tau = \left[\dfrac{\tau_w}{\rho}\right]^{1/2}$, we obtain

$$u_\tau^2 = \chi^2 y^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 \tag{10.34}$$

or

$$\frac{\partial \bar{u}}{\partial y} = \frac{u_\tau}{\chi y} \tag{10.35}$$

On integrating we find

$$\bar{u} = \frac{u_\tau}{\chi} \ln y + C \qquad (10.36)$$

Despite the fact that Eq. (10.36) is derived on the basis of the friction velocity in the neighbourhood of the wall because of the assumption that $\tau_w = \tau_t = $ constant, we shall use it for the entire region. At $y = h$ (at the horizontal mid plane of the channel), we have $\bar{u} = U_{max}$. The constant of integration is eliminated by considering

$$U_{max} = \frac{u_\tau}{\chi} \ln h + C$$

or $$C = U_{max} - \frac{u_\tau}{\chi} \ln h$$

Substituting C in Eq. (10.36), we get

and $$\frac{U_{max} - \bar{u}}{u_\tau} = \frac{1}{\chi} \ln \left(\frac{h}{y} \right) \qquad (10.37)$$

Equation (10.37) is known as universal velocity defect law of Prandtl and its distribution has been shown in Fig. 10.9.

Here, we have seen that the friction velocity u_τ is a reference parameter for velocity. We shall now discuss the problem with $(\bar{u}/u_\tau)$ and η $(= y\, u_\tau/v)$ as parameters. Equation (10.36) can be rewritten once again for this purpose as

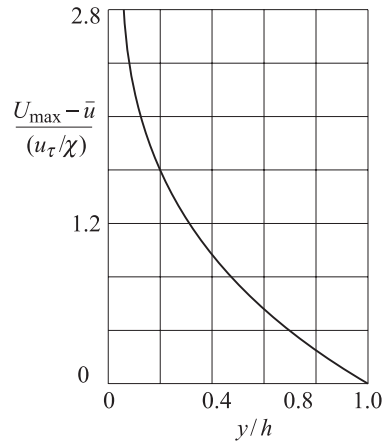$$\frac{\bar{u}}{u_\tau} = \frac{1}{\chi} \ln y + C$$

Fig. 10.9 Distribution of universal velocity defect law of Prandtl in a turbulent channel flow

The no-slip condition at the wall cannot be satisfied with a finite constant of integration. This is expected that the appropriate condition for the present problem

should be that $\bar{u} = 0$ at a very small distance $y = y_0$ from the wall. Hence, Eq. (10.36) becomes

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\chi}\,(\ln y - \ln y_0) \tag{10.38}$$

The distance y_0 is of the order of magnitude of the thickness of the viscous layer. Now we can write Eq. (10.38) as

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\chi}\left[\ln y\frac{u_\tau}{\nu} - \ln \beta\right]$$

or

$$\frac{\bar{u}}{u_\tau} = A_1 \ln \eta + D_1 \tag{10.39}$$

where $A_1 = (1/\chi)$, the unknown β is included in D_1.

   Equation (10.39) is generally known as the *universal velocity* profile because of the fact that it is applicable from moderate to a very large Reynolds number. However, the constants A_1 and D_1 have to be found out from experiments. The aforesaid profile is not only valid for channel (rectangular) flows, it retains the same functional relationship for circular pipes as well. It may be mentioned that even without the assumption of having a constant shear stress throughout, the universal velocity profile can be derived. Interested readers are referred to Example 10.3.

   Experiments, performed by J. Nikuradse, showed that Eq. (10.39) is in good agreement with experimental results. Based on Nikuradse's and Reichardt's experimental data, the empirical constants of Eq. (10.39) can be determined *for a smooth pipe as*

$$\frac{\bar{u}}{u_\tau} = 2.5 \ln \eta + 5.5 \tag{10.40}$$

This velocity distribution has been shown through curve (b) in Fig. 10.10.
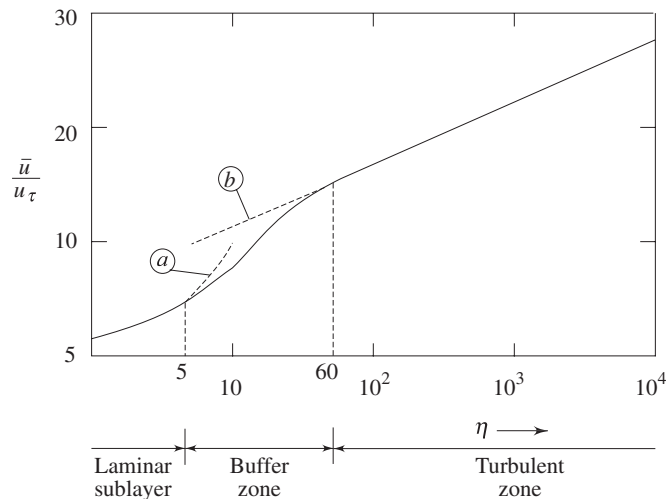

Fig. 10.10   The universal velocity distribution law for smooth pipes

However, the corresponding friction factor concerning Eq. (10.40) is

$$\frac{1}{\sqrt{f}} = 2.0 \log_{10} (\text{Re} \sqrt{f}) - 0.8 \qquad (10.41)$$

As mentioned earlier, the universal velocity profile does not match very close to the wall where the viscous shear predominates the flow. However, von Karman suggested a modification for the *laminar sublayer* and the *buffer zone* which are

$$\frac{\bar{u}}{u_\tau} = \eta = \frac{u_\tau y}{\nu} \qquad \text{for } \eta < 5.0 \qquad (10.42)$$

and

$$\frac{\bar{u}}{u_\tau} = 11.5 \log_{10} \frac{u_\tau y}{\nu} - 3.0 \qquad \text{for } 5 < \eta < 60 \qquad (10.43)$$

Equation (10.42) has been shown through curve (a) in Fig. 10.10.

It may be worthwhile to mention here that a surface is said to be hydraulically smooth so long

$$0 \le \frac{\varepsilon_p u_\tau}{\nu} \le 5 \qquad (10.44)$$

where ε_p is the average height of the protrusions inside the pipe.

Physically, the above expression means that for smooth pipes protrusions will not be extended outside the laminar sublayer. If protrusions exceed the thickness of laminar sublayer, it is conjectured (also justified though experimental verification) that some additional frictional resistance will contribute to pipe friction due to the form drag experienced by the protrusions in the boundary layer. In rough pipes experiments indicate that the velocity profile may be expressed as:

$$\frac{\bar{u}}{u_\tau} = 2.5 \ln \frac{y}{\varepsilon_p} + 8.5 \qquad (10.45)$$

At the centre-line, the maximum velocity is expressed as

$$\frac{U_{\max}}{u_\tau} = 2.5 \ln \frac{R}{\varepsilon_p} + 8.5 \qquad (10.46)$$

Note that ν no longer appears with R and ε_p. This means that for completely rough zone of turbulent flow, *the profile is independent of Reynolds number and a strong function of pipe roughness*. However, for pipe roughness of varying degrees, the recommendation due to Colebrook and White works well. Their formula is

$$\frac{1}{\sqrt{f}} = 1.74 - 2.0 \log_{10} \left[\frac{\varepsilon_p}{R} + \frac{18.7}{\text{Re}\sqrt{f}} \right] \qquad (10.47)$$

where R is the pipe radius.

For $\varepsilon_p \to 0$, this equation produces the result of the smooth pipes (Eq. (10.41)). For $\text{Re} \to \infty$, it gives the expression for friction factor for a completely rough pipe at a very high Reynolds number which is given by

$$f = \frac{1}{\left(2 \log \dfrac{R}{\varepsilon_p} + 1.74\right)^2} \tag{10.48}$$

Turbulent flow through pipes has been investigated by many researchers because of its enormous practical importance.

In the next section, we shall discuss, in detail the velocity distribution and other important aspects of turbulent pipe flows.

10.11 FULLY DEVELOPED TURBULENT FLOW IN A PIPE FOR MODERATE REYNOLDS NUMBERS

The entry length of a turbulent flow is much shorter than that of a laminar flow, J. Nikuradse determined that a fully developed profile for turbulent flow can be observed after an entry length of 25 to 40 diameters. We shall focus herein our attention to fully developed turbulent flow. Considering a fully developed turbulent pipe flow (Fig. 10.11) we can write

$$2 \pi R \, \tau_w = -\left(\frac{dp}{dx}\right) \pi R^2 \tag{10.49}$$

or $\qquad \left(-\dfrac{dp}{dx}\right) = \dfrac{2\tau_w}{R} \tag{10.50}$
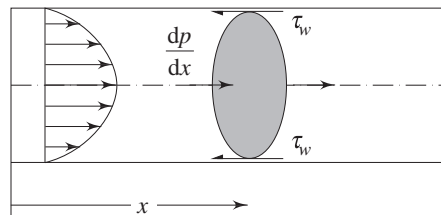
Fig. 10.11 Fully developed turbulent pipe flow

It can be said that in a fully developed flow, the pressure gradient balances the wall shear stress only and has a constant value at any x. However, the friction factor (Darcy friction factor) is defined in a fully developed flow as

$$-\left(\frac{dp}{dx}\right) = \frac{\rho f U_{av}^2}{2D} \tag{10.51}$$

Comparing Eq. (10.50) with Eq. (10.51), we can write

$$\tau_w = \frac{f}{8} \rho U_{av}^2 \tag{10.52}$$

H. Blasius conducted a critical survey of available experimental results and established the empirical correlation for the above equation as

$$f = 0.3164 \, \text{Re}^{-0.25}, \text{ where } \text{Re} = \rho U_{av} D/\mu \tag{10.53}$$

It is found that the Blasius's formula is valid in the range of Reynolds number of $\text{Re} \leq 10^5$. At the time when Blasius compiled the experimental data, results for higher Reynolds numbers were not available. However, later on, J. Nikuradse carried out experiments with the laws of friction in a very wide range of Reynolds numbers, $4 \times 10^3 \leq \text{Re} \leq 3.2 \times 10^6$. The velocity profile in this range follows:

$$\frac{u}{\overline{u}} = \left[\frac{y}{R} \right]^{1/n} \tag{10.54}$$

where $\overline{u}$ is the time mean velocity at the pipe centre and y is the *distance from the wall*. The exponent n varies slightly with Reynolds number. In the range of $\text{Re} \sim 10^5$, n is 7.

The ratio of $\overline{u}$ and U_{av} for the aforesaid profile is found out by considering the volume flow rate Q as

$$Q = \pi R^2 U_{av} = \int_0^R 2\pi r\, u\, dr$$

or
$$\pi R^2 U_{av} = 2\pi\, \overline{u} \int_R^0 (R - y)\, (y/R)^{1/n}\, (-dy)$$

or
$$\pi R^2\, U_{av} = 2\pi\, \overline{u} \left[\frac{n}{n+1} \left(R^{\frac{n-1}{n}} y^{\frac{n+1}{n}} \right) - \frac{n}{2n+1} \left(y^{\frac{2n+1}{n}} R^{-\frac{1}{n}} \right) \right]_0^R$$

or
$$\pi R^2\, U_{av} = 2\pi\, \overline{u} \left[R^2\, \frac{n}{n+1} - \frac{n}{2n+1} R^2 \right]$$

or
$$\pi R^2\, U_{av} = 2\pi R^2\, \overline{u} \left[\frac{n^2}{(n+1)(2n+1)} \right]$$

or
$$\frac{U_{av}}{\overline{u}} = \frac{2\, n^2}{(n+1)(2n+1)} \tag{10.55a}$$

Now, for different values of n (for different Reynolds numbers) we shall obtain different values of $U_{av}/\overline{u}$ from Eq. (10.55a). On substitution of Blasius resistance formula (10.53) in Eq. (10.52), the following expression for the shear stress at the wall can be obtained.

$$\tau_w = \frac{0.3164}{8}\, \text{Re}^{-0.25}\, \rho U_{av}^2$$

or
$$\tau_w = 0.03955\, \rho U_{av}^2 \left(\frac{\nu}{2R\, U_{av}} \right)^{1/4}$$

or
$$\tau_w = 0.03325\, \rho U_{av}^{7/4} \left(\frac{\nu}{R} \right)^{1/4}$$

or
$$\tau_w = 0.03325\, \rho \left(\frac{U_{av}}{\overline{u}} \right)^{7/4} (\overline{u})^{7/4} \left(\frac{\nu}{R} \right)^{1/4}$$

For $n = 7$, $U_{av}/\overline{u}$ becomes equal to 0.8. Substituting $U_{av}/\overline{u} = 0.8$ in the above equation, we get

$$\tau_w = 0.03325 \, \rho \, (0.8)^{1/4}(\overline{u})^{7/4} \, (v/R)^{1/4}$$

Finally it produces $\quad \tau_w = 0.0225 \, \rho \, (\overline{u})^{7/4} \, (v/R)^{1/4}$ \hfill (10.55b)

or $\qquad\qquad u_\tau^2 = 0.0225 \, (\overline{u})^{7/4} \left(\dfrac{v}{R}\right)^{1/4}$

where u_τ is friction velocity. However, u_τ^2 may be splitted into $u_\tau^{7/4}$ and $u_\tau^{1/4}$ and we obtain

$$\left(\frac{\overline{u}}{u_\tau}\right)^{7/4} = 44.44 \left(\frac{u_\tau R}{v}\right)^{1/4}$$

or $\qquad\qquad \dfrac{\overline{u}}{u_\tau} = 8.74 \left(\dfrac{u_\tau R}{v}\right)^{1/7}$ \hfill (10.56a)

Now we can assume that the above equation is not only valid at the pipe axis ($y = R$) but also at any distance from the wall y and a general form is proposed as

$$\frac{\overline{u}}{u_\tau} = 8.74 \left(\frac{y u_\tau}{v}\right)^{1/7} \tag{10.56b}$$

In conclusion, it can be said that (1/7)th power velocity distribution law (10.56b) can be derived from Blasius's resistance formula (10.53). Equation (10.55b) gives the shear stress relationship in pipe flow at a moderate Reynolds number, i.e $Re \leq 10^5$. Unlike very high Reynolds number flow, here laminar effect cannot be neglected and the laminar sublayer brings about remarkable influence on the outer zones.

It is worth mentioning that the friction factor for pipe flows, f, defined by Eq. (10.53) is valid for a specific range of Reynolds number and for a particular surface condition. The experimental results for a wide range of Reynolds numbers and variety of pipe roughness can be summarized through *Moody diagram* which has been shown in Chapter 11.

10.12 SKIN FRICTION COEFFICIENT FOR BOUNDARY LAYERS ON A FLAT PLATE

Calculations of skin friction drag on lifting surface and on aerodynamic bodies are somewhat similar to the analyses of skin friction on a flat plate. Because of zero pressure gradient, the flat plate at zero incidence is easy to consider. In some of the applications cited above, the pressure gradient will differ from zero but the skin friction will not be dramatically different so long there is no separation.

We begin with the momentum integral equation for flat plate boundary layer which is valid for both laminar and turbulent flow.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(U_\infty^2 \, \delta^{**}\right) = \frac{\tau_w}{\rho} \tag{10.57a}$$

Invoking the definition of C_{fx} $\left(C_{fx} = \dfrac{\tau_w}{\dfrac{1}{2}\rho U_\infty^2} \right)$, Eq. (10.57a) can be rewritten as

$$C_{fx} = 2\,\frac{\mathrm{d}\delta^{**}}{\mathrm{d}x} \tag{10.57b}$$

Due to the similarity in the laws of wall, correlations of previous section may be applied to the flat plate by substituting δ for R and U_∞ for the time mean velocity at the pipe centre. The rationale for using the turbulent pipe flow results in the situation of a turbulent flow over a flat plate is to consider that the time mean velocity, at the centre of the pipe is analogous to the free stream velocity, both the velocities being defined at the edge of boundary layer thickness.

Finally, the velocity profile will be [following Eq. (10.54)]

$$\frac{u}{U_\infty} = \left[\frac{y}{\delta}\right]^{1/7} \qquad \text{for Re} \le 10^5 \tag{10.58}$$

If we evaluate momentum thickness with this profile, we shall obtain

$$\delta^{**} = \int_0^\delta \left(\frac{y}{\delta}\right)^{1/7}\left[1-\left(\frac{y}{\delta}\right)^{1/7}\right]\mathrm{d}y = \frac{7}{72}\,\delta \tag{10.59}$$

Consequently, the law of shear stress (in range of Re $\le 10^5$) for the flat plate is found out by making use of the pipe flow expression of Eq. (10.55b) as

$$\tau_w = 0.0225\,\rho(\bar{u})^{7/4}\left(\frac{\nu}{R}\right)^{1/4}$$

or $\qquad \dfrac{\tau_w}{\rho(\bar{u})^2} = 0.0225\left[\dfrac{\nu}{R\bar{u}}\right]^{1/4}$

Substituting U_∞ for $\bar{u}$ and δ for R in the above expression, we get

or $\qquad \dfrac{\tau_w}{\rho U_\infty^2} = 0.0225\left[\dfrac{\nu}{\delta U_\infty}\right]^{1/4} \tag{10.60}$

Once again substituting Eqs (10.59) and (10.60) in Eq. (10.57), we obtain

$$\frac{7}{72}\cdot\frac{\mathrm{d}\delta}{\mathrm{d}x} = 0.0225\left[\frac{\nu}{\delta U_\infty}\right]^{1/4}$$

or $\qquad \delta^{1/4}\dfrac{\mathrm{d}\delta}{\mathrm{d}x} = 0.2314\left[\dfrac{\nu}{U_\infty}\right]^{1/4}$

or $\qquad \delta^{5/4} = 0.2892\,x\left(\dfrac{\nu}{U_\infty}\right)^{1/4} + C \tag{10.61}$

For simplicity, if we assume that the turbulent boundary layer grows from the leading edge of the plate we shall be able to apply the boundary conditions $x = 0$, $\delta = 0$ which will yield $C = 0$, and Eq. (10.61) will become

$$(\delta/x)^{5/4} = 0.2892 \left[\frac{\nu}{xU_\infty} \right]^{1/4}$$

or

$$\frac{\delta}{x} = 0.37 \left[\frac{\nu}{xU_\infty} \right]^{1/5}$$

or

$$\frac{\delta}{x} = 0.37 \, (\mathrm{Re}_x)^{-1/5} \tag{10.62}$$

where

$$\mathrm{Re}_x = (U_\infty x)/\nu$$

From Eqs (10.57b), (10.59) and (10.62), it is possible to calculate the average skin friction coefficient on a flat plate as

$$\overline{C}_f = 0.072 \, (\mathrm{Re}_L)^{-1/5} \tag{10.63}$$

It can be shown that Eq. (10.63) predicts the average skin friction coefficient correctly in the regime of Reynolds number below 2×10^6.

This result is found to be in good agreement with the experimental results in the range of Reynolds number between 5×10^5 and 10^7 which is given by

$$\overline{C}_f = 0.074 \, (\mathrm{Re}_L)^{-1/5} \tag{10.64}$$

Equation (10.64) is a widely accepted correlation for the average value of turbulent skin friction coefficient on a flat plate.

With the help of Nikuradse's experiments, Schlichting obtained the semi-empirical equation for the average skin friction coefficient as

$$\overline{C}_f = \frac{0.455}{(\log \mathrm{Re})^{2.58}} \tag{10.65}$$

Equation (10.65) was derived assuming the flat plate to be completely turbulent over its entire length. In reality, a portion of it is laminar from the leading edge to some downstream position. For this purpose, it was suggested to use

$$\overline{C}_f = \frac{0.455}{(\log \mathrm{Re})^{2.58}} - \frac{A}{\mathrm{Re}} \tag{10.66a}$$

where A has various values depending on the value of Reynolds number at which the transition takes place. If the trasition is assumed to take place around a Reynolds number of 5×10^5, the average skin friction correlation of Schlichling can be written as

$$\overline{C}_f = \frac{0.455}{(\log \mathrm{Re})^{2.58}} - \frac{1700}{\mathrm{Re}} \tag{10.66b}$$

All that we have presented so far, are valid for a smooth plate. Schlichting used a logarithmic expression for turbulent flow over a rough surface and derived

$$\overline{C}_f = \left(1.89 + 1.62 \log \frac{L}{\varepsilon_p} \right)^{-2.5} \tag{10.67}$$

Summary

- Turbulent motion is an irregular motion of fluid particles in a flow field. However, for homogeneous and isotropic turbulence, the flow field can be described by time-mean motions and fluctuating components. This is called Reynolds decomposition of turbulent flow.

- In a three dimensional flow field, the velocity components and the pressure can be expressed in terms of the time-averages and the corresponding fluctuations. Substitution of these depenent variables in the Navier–Stokes equations for incompressible flow and subsequent time averaging yield the governing equations for the turbulent flow. The mean velocity components of turbulent flow satisfy the same Navier-Stokes equations for laminar flow. However, for the turbulent flow, the laminar stresses are increased by additional stresses arising out of the fluctuating velocity components. These additional stresses are known as apparent stresses of turbulent flow or Reynolds stresses.

- In analogy with the laminar shear stresses, the turbulent shear stresses can be expressed in terms of mean velocity gradients and a mixing coefficient known as eddy viscosity. The eddy viscosity (v_t) can be expressed as $v_t = l^2 \left| \dfrac{d\bar{u}}{dy} \right|$, where l is known as Prandtl's mixing length.

- For a homogeneous and isotropic turbulence, most widely used value of mixing length is given by $l = \mathcal{X}\, y$. In this expression, y is the distance from the wall and $\mathcal{X}$ is known as von Karman constant (≈ 0.4). For high Reynolds number, fully developed turbulent duct flows, the velocity profile is given by

$$\frac{\bar{u}}{u_\tau} = A_1 \ln \eta + D_1$$

where $\bar{u}$ is the time-mean velocity at any $\eta\,(=yu_\tau/v)$ and u_τ is the friction velocity given by $\sqrt{\tau_w/\rho}$. The constants A_1 and D_1 are determined from experiments which are 2.5 and 5.5, respectively, for smooth pipes. The corresponding friction factor (f) is given by

the expression $\dfrac{1}{\sqrt{f}} = 2.0 \log_{10} (\mathrm{Re}\,\sqrt{f}) - 0.8$.

- However, for pipe roughness of varying degree, the following recommendation of Colebrook and White works well

$$\frac{1}{\sqrt{f}} = 1.74 - 2.0 \log_{10} \left[\frac{\varepsilon_p}{R} + \frac{18.7}{\mathrm{Re}\,\sqrt{f}} \right]$$

where ε_p/R is pipe roughness.

- In the range of $\mathrm{Re} \leq 10^5$, the velocity distribution in a smooth pipe is given by

$$\frac{\bar{u}}{u_\tau} = 8.74 \left(\frac{y u_\tau}{\nu}\right)^{1/7}$$

- The friction factor in this regime is given by Blasius as

$$f = 0.3164 \ (\text{Re})^{-0.25}$$

- The growth of boundary layer for turbulent flow over a flat plate is given by

$$\frac{\delta}{x} = 0.37 \ (\text{Re}_x)^{-1/5}$$

- The expression for the average skin friction coefficient on the entire plate of length L has been determined as

$$\overline{C}_f = 0.072 \ (\text{Re}_L)^{-1/5}$$

- This result is found to be in good agreement with the experimental results in the range of $5 \times 10^5 < \text{Re} < 10^7$ which is given by

$$\overline{C}_f = 0.074 \ (\text{Re}_L)^{-1/5}$$

- For turbulent flow over a rough plate, the average skin friction coefficient is given by

$$\overline{C}_f = \left(1.89 + 1.62 \log \frac{L}{\varepsilon_p}\right)^{-2.5}$$

References

1. Tennekes, H., and Lumley, J.L, *A First Course in Turbulence*, the MIT Press, Combridge, Massachusetts, 1972.
2. Hinze, J.O., *Turbulence*, McGraw-Hill Book Company, New York, 1987.

Solved Examples

Example 10.1   Prove that

$$\overline{\int_{t-T/2}^{t+T/2} \phi \, d\xi} = \int_{t-T/2}^{t+T/2} \overline{\phi} \, d\xi$$

where ϕ is a constinuous function of ξ

Solution

$$\overline{\int_{t-T/2}^{t+T/2} \phi \, d\xi} = \frac{1}{T} \int_{t-T/2}^{t+T/2} \left[\int_{t-T/2}^{t+T/2} \phi \, d\xi\right] dt$$

Changing the order of integration, we can write

or

$$\overline{\int_{t-T/2}^{t+T/2} \phi \, d\xi} = \int_{t-T/2}^{t+T/2} \frac{1}{T} \left[\int_{t-T/2}^{t+T/2} \phi \, dt \right] d\xi$$

or

$$\overline{\int_{t-T/2}^{t+T/2} \phi \, d\xi} = \int_{t-T/2}^{t+T/2} \overline{\phi} \, d\xi$$

Example 10.2 The well known scientist *Theodore von Karman* suggested the mixing length to be $l = \chi \left| \dfrac{d\overline{u}/dy}{d^2\overline{u}/dy^2} \right|$. Using this relation drive the velocity profile near the wall of a flat-plate boundary layer flow.

Solution   We know

$$\tau_t = \mu_t \frac{\partial \overline{u}}{\partial y}, \text{where } \mu_t = \rho l^2 \left| \frac{\partial \overline{u}}{\partial y} \right|$$

So,

$$\tau_t = \rho l^2 \left| \frac{\partial \overline{u}}{\partial y} \right|^2$$

Substituting *von Karman's* suggestion, we get

$$\tau_t = \frac{\rho \chi^2 (d\overline{u}/dy)^2 (d\overline{u}/dy)^2}{(d^2\overline{u}/dy^2)^2} = \frac{\rho \chi^2 (d\overline{u}/dy)^4}{(d^2\overline{u}/dy^2)^2}$$

or

$$\left(\frac{d\overline{u}}{dy} \right)^4 = \frac{\tau_t}{\rho} \frac{1}{\chi^2} \left(\frac{d^2\overline{u}}{dy^2} \right)^2$$

Assuming $\tau_t = \tau_w$ and considering $u_\tau = \sqrt{\tau_w/\rho}$

$$\left(\frac{d\overline{u}}{dy} \right)^4 = \frac{u_\tau^2}{\chi^2} \left(\frac{d^2\overline{u}}{dy^2} \right)^2$$

Taking the square root, and applying physical argument that $(d\overline{u}/dy)$ cannot be imaginary, we obtain

$$\left(\frac{d\overline{u}}{dy} \right)^2 = \pm \frac{u_\tau}{\chi} \left(\frac{d^2\overline{u}}{dy^2} \right)$$

Let

$$m = d\overline{u}/dy$$

then

$$dm/dy = \pm \frac{\chi}{u_\tau} m^2$$

Integration yields,

$$-\frac{1}{m} = \pm \frac{\chi}{u_\tau} y + C_1$$

Using $m \Rightarrow \infty$ as $y \Rightarrow 0$, we get $C_1 = 0$

Then,
$$\frac{d\bar{u}}{dy} = \frac{u_\tau}{\chi y}, \qquad \text{since } \frac{d\bar{u}}{dy} \geq 0$$

Integrating, we obtain,

$$\bar{u} = \frac{u_\tau}{\chi} \ln y + C_2$$

At some value $\qquad y = y_0$, $\bar{u} = 0$

Invoking this, $\qquad C_2 = -\frac{u_\tau}{\chi} \ln y_0$

Thus, $\qquad \dfrac{\bar{u}}{u_\tau} = \dfrac{1}{\chi} \cdot \ln (y - y_0)$

Let us substitute $y_0 = \beta \dfrac{\nu}{u_\tau}$ order of which is same as viscous sublayer and β is an arbitrary constant.

We shall get, thus,

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\chi} \left(\ln \frac{u_\tau y}{\nu} - \ln \beta \right)$$

or
$$\frac{\bar{u}}{u_\tau} = A_1 \ln \eta + D_1$$

This is the universal velocity profile.

Example 10.3 Using *Karman's* relation $l = \chi \left| \dfrac{d\bar{u}/dy}{d^2\bar{u}/dy^2} \right|$, show that the universal velocity distribution in a fully developed channel flow (Fig. 10.11) is given by

$$\frac{U_{\max} - \bar{u}}{u_\tau} = -\frac{1}{\chi} \left[\ln \left(1 - \sqrt{\frac{y}{h}} \right) + \sqrt{\frac{y}{h}} \right]$$

where, $2h$ is the height of the channel, y is the distance measured from the centre line of the channel and χ is an empirical constant. The pressure gradient in flow direction is $-(dp/dx)$.

Solution From Reynolds equation, we get

$$\frac{\partial(\tau_t)}{\partial y} = \frac{\partial \bar{p}}{\partial x}$$

or
$$\tau_t = \left(\frac{\partial \bar{p}}{\partial x} \right) y + C_1$$

At $y = 0$, $\tau_t = 0$, that makes $C_1 = 0$

Thus,
$$\tau_t = \left(\frac{\partial \bar{p}}{\partial x} \right) y$$

and
$$\tau_w = \left(\frac{\partial \bar{p}}{\partial x}\right) h$$

we get
$$\frac{\tau_t}{\tau_w} = \frac{y}{h}$$

From *Karman's* relation, we can write

$$\tau_t = \frac{\rho \chi^2 (\partial \bar{u}/\partial y)^4}{(\partial^2 \bar{u}/\partial y^2)^2}$$

Then
$$\tau_w = \frac{h \rho \chi^2 (\partial \bar{u}/\partial y)^4}{y (\partial^2 \bar{u}/\partial y^2)^2}$$

$$u_\tau^2 = \frac{h \chi^2 (\partial \bar{u}/\partial y)^4}{y (\partial^2 \bar{u}/\partial y^2)^2}$$

Thus,
$$\frac{\partial^2 \bar{u}}{\partial y^2} = \pm \frac{\chi}{u_\tau} \sqrt{\frac{h}{y}} \left(\left|\frac{\partial \bar{u}}{\partial y}\right|\right)^2$$

Substituting for $m = \dfrac{\partial \bar{u}}{\partial y}$ and integrating,

$$-\frac{1}{m} = \pm 2 \frac{\chi}{u_\tau} \sqrt{hy} + C_2$$

at $y = h$, $m \to \infty$ and $C_2 = \pm 2 \dfrac{\chi}{u_\tau} h$

Now, we know that $\dfrac{\partial \bar{u}}{\partial y} \leq 0$ for $y > 0$ and we write

$$\mathrm{d}\bar{u} = -\frac{u_\tau}{2 \chi h} \int \frac{\mathrm{d}y}{1 - \sqrt{y/h}}$$

Substituting for $\quad \xi = 1 - \sqrt{\dfrac{y}{h}}$ and integrating,

$$\bar{u} = \frac{u_\tau}{\chi} [\ln \xi - \xi] + C_3$$

or
$$\bar{u} = \frac{u_\tau}{\chi} \left[\ln\left(1 - \sqrt{\frac{y}{h}}\right) - \left(1 - \sqrt{\frac{y}{h}}\right)\right] + C_3$$

at $y = 0$, $\bar{u} = U_{\max}$

$$C_3 = U_{\max} + \frac{u_\tau}{\chi}$$

Finally,
$$\frac{U_{\max} - \bar{u}}{u_\tau} = \frac{1}{\chi}\left[\ln\left(1 - \sqrt{\frac{y}{h}}\right) - \sqrt{\frac{y}{h}}\right]$$

Example 10.4 During flow over a flat plate the laminar boundary layer undergoes a transition to turbulent boundary layer as the flow proceeds in the downstream. It is observed that a parabolic laminar profile is finally changed into a 1/7th power law velocity profile in the turbulent regime. Find out the ratio of turbulent and laminar boundary layers if the momentum flux within the boundary layer remains constant.

Solution Assume width of the boundary layers be a. Then momentum flux is

$$A = \int u \, \rho \, u \, a \, dy = \rho \, U_\infty^2 \, a \, \delta \int \left(\frac{u}{U_\infty}\right)^2 d\eta$$

where $\eta = y/\delta$

For laminar flow, $\dfrac{u}{U_\infty} = 2 \, \eta - \eta^2$

$$A_{\text{lam}} = \rho U_\infty^2 \, a \, \delta_{\text{lam}} \int_0^1 (4\eta^2 - 4\eta^3 + \eta^4) \, d\eta$$

$$= \rho U_\infty^2 \, a \, \delta_{\text{lam}} \left[\frac{4}{3}\eta^3 - \eta^4 + \frac{\eta^5}{5}\right]_0^1$$

$$= \frac{8}{15} \rho U_\infty^2 \, a \, \delta_{\text{lam}}$$

For 1/7th power law turbulent profile,

$$\frac{\bar{u}}{U_\infty} = \eta^{1/7}$$

$$A_{\text{turb}} = \rho U_\infty^2 \, a \, \delta_{\text{turb}} \int_0^1 (\eta^{1/7})^2 \, d\eta$$

$$= \rho U_\infty^2 \, a \, \delta_{\text{turb}} \int_0^1 \eta^{2/7} \, d\eta$$

$$= \rho U_\infty^2 \, a \, \delta_{\text{turb}} \left[\frac{7}{9}\eta^{9/7}\right]_0^1 = \frac{7}{9} \rho U_\infty^2 \, a \, \delta_{\text{turb}}$$

Comparing the momentum fluxes,

$$\frac{\delta_{\text{turb}}}{\delta_{\text{lam}}} = \frac{72}{105}$$

It is to be noted that generally turbulent boundary layer grows faster than the laminar boundary layer when a completely turbulent flow is considered from the leading edge. However the present result is valid at transition for a constant momentum flow.

Example 10.5 Air ($\rho = 1.23$ kg/m^3 and $v = 1.5 \times 10^{-5}$ m^2/s) is flowing over a flat plate. The free stream speed is 15 m/s. At a distance of 1 m from the leading edge, calculate δ and τ_w for (a) completely laminar flow, and (b) completely turbulent flow for a 1/7th power law velocity profile.

Solution   Applying the results developed in Chapters 9 and 10, we can write for parabolic velocity profile (laminar flow)

$$\frac{\delta}{x} = \frac{5.48}{\sqrt{\text{Re}_x}} \quad \text{and} \quad \tau_w = \mu\frac{\partial u}{\partial y}\Big|_{y=0}$$

$$\text{Re}_x = \frac{Ux}{\nu} = \frac{15 \times 1}{1.5 \times 10^{-5}} = 1.0 \times 10^6$$

$$\delta = \frac{5.48}{\sqrt{1.0 \times 10^6}} \times 1\,\text{m} = 5.48\,\text{mm}$$

$$\tau_w = \mu\frac{\partial u}{\partial y}\Big|_{y=0} = \frac{\mu U_\infty}{\delta} \cdot \frac{d}{d\eta}[2\eta - \eta^2]_{\eta=0}$$

$$\tau_w = \frac{2 \times 1.23 \times 1.5 \times 10^{-5} \times 15}{0.00548} = 0.101\,\text{N}/\text{m}^2$$

For turbulent flow,

$$\frac{\delta}{x} = \frac{0.37}{(\text{Re}_x)^{1/5}} \quad \text{(from Eq. 10.62)}$$

or
$$\delta = \frac{0.370}{(1.0 \times 10^6)^{1/5}} \times 1\,\text{m} = 23.34\,\text{mm}$$

or
$$\delta/x = 0.0233$$

$$\tau_w = 0.0225\,\rho U_\infty^2 \left(\frac{\nu}{U_\infty\delta}\right)^{1/4} \quad \text{(from Eq. 10.60)}$$

$$\tau_w = 0.0225 \times 1.23 \times (15)^2 \left(\frac{\nu}{U_\infty x} \cdot \frac{x}{\delta}\right)^{1/4}$$

or
$$\tau_w = 0.0225 \times 1.23 \times (15)^2 \left[\frac{1}{1.0 \times 10^6} \times \frac{1}{0.0233}\right]^{1/4}$$

$$= 0.502\,\text{N}/\text{m}^2$$

Turbulent boundary layer has a larger shear stress than the laminar boundary layer.

Exercises

10.1   Only write down the option (true/false) or the choice (a, b, c or d) or the appropriate conditions.

     (i)   For flow through pipes, due to the same pressure gradient, the turbulent velocity profile will be more uniform than the laminar velocity profile.

                                                (True/False)

     (ii)   If the mean velocity has a gradient, the turbulence is called isotropic.

                                                (True/False)

     (iii) $\dfrac{\partial u'}{\partial x} + \dfrac{\partial v'}{\partial y} = 0$ for a turbulent flow signifies

(a) conservation bulk momentum transport
(b) increase in u' in positive x direction will be followed by increase in v' in negative y direction
(c) turbulence is anisotropic
(d) turbulence is isotropic

(vi) In a turbulent pipe flow the initiation of turbulence is usually observed at a Reynolds number (based on pipe diameter) of
(a) 3.5×10^5 (b) 2×10^6
(c) between 2000 and 2700   (d) 5000

(v) A turbulent boundary is thought to be comprising of laminar sublayer, a buffer layer and a turbulent zone. The velocity profile outside the laminar sublayer is described by a
(a) parabolic profile   (b) cubic profile
(c) linear profile   (d) logarithmic profile

(vi) A laminar boundary layer is less likely to separate than a turbulent boundary layer.   (True/False)

10.2 Show that, with the help of both the mixing length hypothesis due to *Prandtl* and mixing length law due to *Karman* (given in Example 10.2), the universal velocity profile near the wall in case of a fully developed turbulent flow through a circular pipe can be expressed as

$$\frac{U_{max} - \bar{u}}{u_\tau} = \frac{1}{\kappa} \ln\left(\frac{R}{R-r}\right)$$

where r is the radius of the pipe and κ is a constant.

10.3 Calculate power required to move a flat plate, 8 m long and 3 m wide in water ($\rho = 1000$ kg/m^3, $\mu = 1.02 \times 10^{-3}$ kg/ms) at 8 m/s for the following cases:
(a) the boundary layer is turbulent over the entire surface of the plate
(b) the transition takes place at Re $= 5 \times 10^5$.

Ans. (a) 12.536×10^3 W (b) 12.518×10^3 W

10.4 The transition Reynolds number in a pipe flow based on U_{av} is approximately 2300. How does this value can be extrapolated for the flow over a flat-plate if U_∞ in the flat-plate case is analogous to U_{max} in the pipe and δ is analogous to pipe radius R?

Ans. (Re$_x$ = 2.116×10^5)

10.5 A plate 50 cm long and 2.5 m wide moves in water at a speed of 15 m/s. Estimate its drag if the transition takes place at Re $= 5 \times 10^5$ for (a) a smooth wall, and (b) a rough wall, $\varepsilon_p = 0.1$ mm. For water, $\rho = 1000$ kg/m^3 and $\mu = 1.02 \times 10^{-3}$ kg/ms.

Ans. ((a) 411.405 W (b) 806.168 W)

10.6 In turbulent flat-plate flow, the wall shear stress is given by the formula

$$\tau_w = 0.0225 \ \rho U_\infty^2 \left[\frac{\nu}{\delta U_\infty}\right]^{1/4}$$

Two important equations concerning 1/7th power law velocity profiles are

$$C_{fx} = 2 \ \frac{d\delta^{**}}{dx} \quad \text{and} \quad \delta^{**} = \frac{7}{72}\delta$$

From the above three equations, find the final expression for skin friction coefficient (C_{fx}).

Ans. ($C_{fx} = 0.0576 \ (\text{Re}_x)^{-1/5}$)

10.7 Water flows at a rate of 0.05 m³/s in a 20 cm diameter cast iron pipe ($\varepsilon_p/D = 0.0007$). What is the head (pressure) loss per kilometer of pipe? For water, $\rho = 1000$ kg/m³, $v = 1.0 \times 10^{-6}$ m²/s. Use Moody's Chart.

Ans. (12.2 m)

10.8 Air ($\rho = 1.2$ kg/m³ and $v = 1.5 \times 10^{-5}$ m²/s) flows at a rate of 2.5 m³/s in a 30 cm × 60 cm steel rectangular duct ($\varepsilon_p = 4.6 \times 10^{-5}$ m). What is the pressure drop per 50 m of the duct? Use Moody's chart.

Ans. (217 pa)

10.9 Water is being transported through a rough pipe line ($u_\tau \varepsilon_p /v = 100$), 1 km long with maximum velocity of 4 m/s. If the Reynolds number is 1.5×10^6, find out the diameter of the pipe and power required to maintain the flow. For water, $\rho = 1000$ kg/m³, $v = 1.0 \times 10^{-6}$ m²/s.

Ans. ($D = 0.454$ m, $P = 134.113$ kW)

10.10 Modify the friction drag coefficient given by Eq. (10.64) as $\overline{C}_f = 0.074\,(\text{Re}_L)^{-1/5}$ $- A/\text{Re}_L$. Let the flow be laminar up to a distance X_{cr} from the leading edge and turbulent for $X_{cr} \leq x \leq L$. Consider the transition to occurs at $\text{Re}_x = 5 \times 10^5$.

Ans. ($A = 1700$)

10.11 Air flows over a smooth flat plate at a velocity of 4.4 m/s. The density of air is 1.029 kg/m³ and the kinematic viscosity is 1.35×10^{-5} m²/s. The length of the plate is 12 m in the direction of flow. Calculate (a) the boundary layer thickness at 16 cm and 12 m respectively, from the leading edge and (b) the drag coefficient for the entire plate surface (one side) considering turbulent flow.

Ans. ((a) 3.5×10^{-3} m, and 0.0207 m (b) $\overline{C}_f = 3.554 \times 10^{-3}$)

10.12 The velocity distribution for a laminar boundary layer flow is given by $\dfrac{u}{u_e} = \sin\left(\dfrac{\pi}{2}\cdot\dfrac{y}{\delta}\right)$. The velocity at $y = k$ is given by u_k. It is assumed that the small roughness of height k will not generate eddies to disturb the boundary layer if $\dfrac{u_k k}{v}$ is less than about 5.0. Show that at a distance x from the leading edge, the maximum permissible roughness height for the boundary layer to remain undistributed is given by $\dfrac{k}{c} = \dfrac{A}{(\text{Re})^{3/4}}\left(\dfrac{x}{c}\right)^{1/4}$ where, $\text{Re} = \dfrac{u_e c}{v}$, c is the total length of the plate and A is a constant.